Deterministic Mechanism Design*

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October 14, 2022

Abstract

This paper studies mechanism design in environments where a designer can only commit to deterministic mechanisms. If there is one agent, stochastic mechanisms may strictly dominate deterministic mechanisms. The main theorem shows an equivalence between stochastic and deterministic mechanisms, whenever there are two or more agents. The equivalence is achieved through an indirect deterministic mechanism. The paper goes on to show a deterministic revelation principle: Under ex-post implementation, direct deterministic mechanisms suffice, provided the set of outcomes is binary.

*I thank Amanda Friedenberg, Andreas Blume, Inga Deimen, and Stanley Reynolds.
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1 Introduction

Mechanism design often utilizes stochastic mechanisms. These are mechanisms whose outcomes depend on both the actions of the participants and the realization of a randomization device. Stochastic mechanisms are undesirable in settings where agents can only observe the realization of the randomization device, but not the randomization device itself. In such settings, the designer may have an incentive to alter the way the mechanism introduces stochasticity. As a consequence, the designer may not be able to commit to the randomization device, even if she can commit to other aspects of the mechanism. For instance, the optimal mechanism may involve a seller taking payment with certainty and randomizing over whether or not to allocate a set of objects. (See Manelli and Vincent [2007].) However, the seller may have an incentive to alter the mechanism's randomization to simply keep the objects.

The lack of commitment to stochastic mechanisms is well understood. For instance, Jehiel et al. [2006] observed that "a stochastic mechanism demands not only that a randomization device be available to the mechanism designer, but also that the outcome of the randomization device be objectively verified." Laffont and Martimort [2002] pointed out that "[e]nsuring this verifiability is a more difficult problem than ensuring that a deterministic mechanism is enforced, because any deviation away from a given randomization can only be statistically detected once a sufficient number of realizations of the contracts have been observed."

Deterministic and stochastic mechanisms are not equivalent in settings with one agent. Manelli and Vincent [2007], Manelli and Vincent [2006], and Pycia [2006] each provide examples where the designer is strictly better off by using a randomization device. In fact, Proposition B.1 in Online Appendix B shows that this holds in a broad class of settings with one agent and three or more outcomes. In such settings, there is always a designer that would be strictly better off with a stochastic mechanism.

This paper shows that the dominance of stochastic mechanisms is particular to the one agent environment. The main theorem states that stochastic and deterministic mechanisms are outcome equivalent in settings with multiple agents. That is, any distribution of outcomes that can be achieved with a stochastic mechanism can also be achieved by a deterministic mechanism. So, there is no loss in only using deterministic mechanisms. Importantly, the constructed deterministic mechanism that achieves this equivalence may not be *direct*, in the sense of the revelation principle. In particular, it may require the agents to report more than simply their types. As Strausz [2003] points out, the deterministic version of the revelation principle does not hold in general.

The paper also provides a sufficient condition for a deterministic version of the revelation principle under ex-post implementation. If the outcome set is binary, a deterministic version of the revelation principle holds. This result is relevant for binary social decision problems with no transfers. Examples include a jury deciding whether a suspect is innocent or guilty, a faculty committee deciding whether or not to tenure a faculty member, etc. An implication of the result is that there is no loss in only using deterministic direct mechanisms.

The results contribute to a recent literature in mechanism design that explores the scope of deterministic mechanisms. For instance, Jehiel et al. [2006] analyzes ex-post implementable direct deterministic mechanisms in settings with a continuum set of types. Jarman and Meisner [2017] analyzes the scope of direct deterministic mechanisms that are implementable in dominant strategies. Chen, He, Li, and Sun [2019] analyzes the scope of Bayesian incentive compatible direct deterministic mechanisms in settings with finite alternatives, independent types, and atomless distributions. The equivalence result here departs from the literature, in the sense that it makes use of *indirect* deterministic mechanisms.

The proof of the equivalence result shows that, when there are two or more agents, the realization of a randomization device can be replicated by agents' behavior in a particular indirect mechanism. This step is reminiscent of a literature in repeated games, which replicates a public correlation device by having players use jointly controlled lotteries. (See. e.g., Aumann, Maschler, and Stearns [1968] and Fudenberg and Maskin [1991].) Because the structure of repeated incentives are different from the incentives here, the nature of the proofs differ.

2 Model

Through this paper take the following conventions. Endow a Polish space Ω with the Borel sigma algebra. Denote by $\Delta(\Omega)$ the set of probability measures on Ω and endow $\Delta(\Omega)$ with the topology of weak convergence. Endow the product of topological spaces with the product topology. In addition, for any given set of indices I and family of sets $(\Omega_i)_{i\in I}$ write $\Omega_{-i} = \prod_{j\in I\setminus\{i\}} \Omega_j$ and $\Omega = \prod_{i\in I} \Omega_i$.

2.1 Environment

The set of agents is $I = \{1, ..., n\}$. For each agent i there is a set of types Θ_i which is a compact metric space. The type profile $(\theta_i : i \in I)$ is drawn from a common prior $\mu \in \Delta(\Theta)$ with full support.[1] Each agent i learns $\theta_i \in \Theta_i$, but not $\theta_{-i} \in \Theta_{-i}$. There is a compact metric space of outcomes Y. The Bernoulli utility function of agent i is a continuous function $u_i : \Theta \times Y \to \mathbb{R}$.

The designer does not know the agents' types. She seeks to maximize a continuous function $\pi : \Theta \times Y \to \mathbb{R}$ given the prior μ. Examples of these objectives are revenue maximization, utilitarian welfare, etc.

2.2 Mechanisms

A **mechanism** is a tuple $\mathcal{M} = (R, m)$ described as follows: The set of report profiles R is a product set $\prod_i R_i$, where each R_i is a compact metric set of reports for i. The measurable mapping $m : R \to \Delta(Y)$ is the mechanism's protocol. The mechanism $\mathcal{M} = (R, m)$ corresponds to a game in which agents simultaneously choose reports $r = (r_1, ..., r_n) \in R$ and the mechanism randomly selects an outcome according to $m(r) \in \Delta(Y)$. Call the mechanism $\mathcal{M}$ **deterministic** if, for each profile of reports $r \in R$, the support of $m(r)$ is a singleton.

[1] The results in the paper do not depend on the common prior assumption.

A mixed strategy for i is a measurable map $\sigma_i : \Theta_i \to \Delta(R_i)$. Write $\sigma = (\sigma_1, ..., \sigma_n)$ for a strategy profile. With abuse of notation, let $\sigma(\theta) \in \Delta(R)$ be the product of measures $(\sigma_1(\theta_1),...,\sigma_n(\theta_n)) \in \prod_{i \in I} \Delta(R_i)$. Write

$$\mathbb{E}[u_i \mid \mathcal{M}, \sigma, \theta] = \int_{y \in Y} \int_{r \in R} u_i(\theta, y) \; d\sigma(\theta) \; dm(r),$$

for i's expected utility, under the mechanism $\mathcal{M}$, given a strategy profile σ and a type profile $\theta \in \Theta$. Define $\mathbb{E}[u_i \mid \mathcal{M}, (r_i, \sigma_{-i}), \theta]$ likewise.

Definition 2.1. *A strategy profile σ is an* ***ex-post equilibrium*** *for the mechanism $\mathcal{M}$ if, for each $\theta \in \Theta$ and each report $r_i \in R_i$, $\mathbb{E}[u_i \mid \mathcal{M}, \sigma, \theta] \geq \mathbb{E}[u_i \mid \mathcal{M}, (r_i, \sigma_{-i}), \theta]$.*

In an ex-post equilibrium, each type θ_i's distribution of actions $\sigma_i(\theta_i)$ is optimal given each type profile of the other agents and the behavior specified by σ_{-i}.

The designer's ex-ante payoff from a strategy profile σ under $\mathcal{M}$ is given by

$$\mathbb{E}[\pi \mid \mathcal{M}, \sigma] = \int_{\theta \in \Theta} \int_{r \in R} \int_{y \in Y} \pi(\theta, y) \; dm(r) \; d\sigma(\theta) \; d\mu.$$

The designer's problem is to find a mechanism $\mathcal{M}$ and an ex-post equilibrium σ thereof that maximizes the value $\mathbb{E}[\pi \mid \mathcal{M}, \sigma]$.

Say that a mechanism $\mathcal{M} = (R, m)$ is **direct** if $R = \Theta$. In a direct mechanism, the strategy σ_i^* is **honest** if, for each $\theta_i \in \Theta_i$, $\sigma_i^*(\theta_i)(\theta_i) = 1$. So, under the honest strategy profile, each agent truthfully reports their type. A direct mechanism $\mathcal{M}$ is called **ex-post incentive compatible (EPIC)** if the honest strategy profile is an ex-post equilibrium.

The revelation principle states that, for each ex-post equilibrium of a mechanism, there is an EPIC direct mechanism where the honest strategy profile provides the same distribution of types/outcomes as the ex-post equilibrium; thus, it provides the same payoffs to each agent and the designer. (See Bergemann and Morris [2008] for this formalization.)

3 Stochastic vs. Deterministic Mechanisms

This section explores the scope of stochastic mechanisms versus deterministic mechanisms. (It does not restrict attention to direct deterministic mechanisms.) We show that the key feature is whether or not there are at least two agents.

3.1 Dominance

If there is only one agent, stochastic mechanisms can dominate deterministic mechanisms. This is illustrated by the following example.

Example 3.1. ***Dominance of Stochastic Mechanisms.*** *There is one agent with a set of types $\Theta = \{\underline{\theta}, \overline{\theta}\}$. The common prior assigns $\mu(\overline{\theta}) = \frac{1}{2}$. The set of outcomes is $Y = \{y_1, y_2, y_3\}$. For each $\theta \in \Theta$ and $y_k \in Y$, the utility of the agent is given by $u(\theta, y_k) = k$. The designer's utility is*

$\pi(\theta, y) = 1$ if $(\theta, y) \in \{(\overline{\theta}, y_1), (\underline{\theta}, y_2), (\overline{\theta}, y_3)\}$ and $\pi(\theta, y) = 0$, otherwise. Notice that preferences are state independent for the agent but not for the designer.

There is an EPIC direct mechanism that gives the designer an expected payoff of 1. To see this, let $\mathcal{M} = (\Theta, m)$ be such that (i) $m(\underline{\theta})$ selects y_2 with probability one and (ii) $m(\overline{\theta})$ equally randomizes between outcomes y_1 and y_3. Notice that, for each type, each report provides the agent an expected utility of 2. So, $\mathcal{M}$ is EPIC and gives an expected payoff of 1 to the designer under the honest strategy profile.

By contrast, in each deterministic mechanism (direct or indirect), the designer gets a payoff of $\frac{1}{2}$. To see this, fix a deterministic mechanism $\mathcal{M} = (R, m)$ and write $Y(\mathcal{M}) = \cup_{r \in R} \text{Supp}\,(m(r))$ for the set of reachable outcomes of $\mathcal{M}$. Since $\mathcal{M}$ is deterministic, the agent effectively chooses an outcome $y \in Y(\mathcal{M})$ by choosing its associated report. In any equilibrium, each type selects its optimal outcome in $Y(\mathcal{M})$. Since both types share the same strict preferences, both types select the same outcome $y \in Y(\mathcal{M})$. Thus, the mechanism provides a expected payoff of $\frac{1}{2}$ to the designer.

Online Appendix B extends Example 3.1 to a broad class of settings with one agent and three or more outcomes. In these settings, there is always some objective of the designer for which some stochastic mechanism dominates all deterministic mechanisms.[2]

3.2 Equivalence

In settings with two or more agents, stochastic and deterministic mechanisms are equivalent. The key is that if agents use mixed strategies, the designer can use the realization of the mixed strategies to mimic a randomization device. Importantly, this is possible without giving incentives to manipulate the randomization device or misreport their private information. We now explain how.

Let $n \geq 2$. Fix an EPIC direct mechanism $\mathcal{M} = (\Theta, m)$. We construct an indirect deterministic mechanism that induces the same distribution of outcomes that $\mathcal{M}$ induces.

Write $\lambda \in \Delta([0,1])$ for the Lebesgue probability measure conditional on $[0,1]$. Fix a type profile $\theta \in \Theta$, and consider the probability measure $m(\theta) \in \Delta(Y)$. Lemma A.1 shows that there is a measurable mapping $g_\theta : [0,1] \to Y$ such that, for each measurable $A \subseteq Y$, $m(\theta)(A) = \lambda(g_\theta^{-1}(A))$. (This uses ideas from the inverse transformation method.) So, if the designer can "construct" a random variable $\mathbf{u}$ uniform in [0,1], then she can simulate a randomization device following the distribution $m(\theta)$ by selecting the outcome $g_\theta(\mathbf{u}) \in Y$. We construct a mechanism where $\mathbf{u}$ is derived from the agents' mixing.

Write $\hat{R}_i = \Theta_i \times [0,1]$ for the set of reports for i and write $\hat{R} = \prod_{i \in I} \hat{R}_i$. Define $\hat{y} : \hat{R} \to Y$ so that $\hat{y}(\theta', s) = g_{\theta'}\left(\text{res}\left(\sum_{k=1}^{n} s_k\right)\right)$, where $\text{res}(x) = x - \lfloor x \rfloor$ is the integer residual of $x \in \mathbb{R}$. In the induced deterministic mechanism, after observing θ_i, each agent i reports a type $\theta'_i \in \Theta_i$ and a number $s_i \in [0,1]$, where s_i is a supplemental report. The mechanism observes both the type and supplemental reports, $\theta' = (\theta'_1, ..., \theta'_n)$ and $s = (s_1, ..., s_n)$, and deterministically selects the outcome $\hat{y}(\theta', s)$. Write $\hat{m} : \hat{R} \to \Delta(Y)$ for the mapping such that $\hat{m}(\theta', s)$ assigns probability one

[2]This is different from the classic equivalence result in the problem of selling an indivisible object, which assumes that the objective takes a particular form.

to $\hat{y}(\theta', s)$. Call $\hat{\mathcal{M}} = (\hat{R}, \hat{m})$ the **deterministic mechanism induced by** $\mathcal{M}$. Notice that $\hat{\mathcal{M}}$ is deterministic but not direct.

There is an **honest-uniform strategy** $\hat{\sigma}_i : \Theta_i \to \Delta(\hat{R}_i)$ such that, for each $\theta_i \in \Theta_i$, (1) $\text{marg}_{\Theta_i}\sigma_i(\theta_i)(\theta_i) = 1$ and (2) $\text{marg}_{[0,1]}\hat{\sigma}_i(\theta_i)$ is uniform on $[0,1]$. In the honest-uniform strategy profile $\hat{\sigma} = (\hat{\sigma}_1, ..., \hat{\sigma}_n)$ each agent i truthfully reports her type θ_i and uniformly draws a supplemental report $s_i \in [0,1]$.

It remains to show that $\hat{\sigma}$ is an ex-post equilibrium of $\hat{\mathcal{M}}$ and that $\hat{\sigma}$ is payoff equivalent to the honest strategy profile of $\mathcal{M}$. These will follow from a key property of $\hat{\mathcal{M}}$ and $\hat{\sigma}$: Fix an agent $i \in I$ and a number $c \in \mathbb{R}$. Lemma A.2 shows that if $\mathbf{s_i}$ is a uniform random variable on $[0,1]$, then $\mathbf{u} = \text{res}(c + \mathbf{s_i})$ is also a uniform random variable on $[0,1]$. Different values $c \in \mathbb{R}$ may lead to different realizations of $\mathbf{u}$ but do not change its distribution. So, as long as there is some agent i that selects s_i uniformly on $[0,1]$, then the integer residual of the sum of the supplemental reports is uniform on $[0,1]$—independently of the strategies of the other agents. This property implies the following result.

Lemma 3.1. *Let $n \geq 2$. Fix a direct mechanism $\mathcal{M}$ and let σ^* be the honest strategy profile thereof. If $\hat{\mathcal{M}}$ is a deterministic mechanism induced by $\mathcal{M}$ and $\hat{\sigma}$ is the honest-uniform strategy profile, then*

(i) $\mathbb{E}[\pi \mid \hat{\mathcal{M}}, \hat{\sigma}] = \mathbb{E}[\pi \mid \mathcal{M}, \sigma^*]$, *and*

(ii) for each $(\theta_i', s_i') \in R_i$ *and* $\theta \in \Theta$, $\mathbb{E}[u_i \mid \hat{\mathcal{M}}, ((\theta_i', s_i'), \hat{\sigma}_{-i}), \theta] = \mathbb{E}[u_i \mid \mathcal{M}, (\theta_i', \sigma_{-i}^*), \theta]$.

Part (i) states that the designer gets the same payoffs in $\mathcal{M}$ and its induced deterministic mechanism $\hat{\mathcal{M}}$. Part (ii) states that agent i's payoff under $\hat{\mathcal{M}}$ and $\hat{\sigma}_{-i}$ does not depend on the supplemental report s_i'. Moreover, the agent's payoff of reporting (θ_i', s_i') is given by her payoff of reporting θ_i' on the mechanism $\mathcal{M}$. Lemma 3.1 immediately implies the following result.

Theorem 3.1. *Let $n \geq 2$. Fix an EPIC direct mechanism $\mathcal{M}$ and let σ^* be the honest strategy profile thereof. If $\hat{\mathcal{M}}$ is a deterministic mechanism induced by $\mathcal{M}$ and $\hat{\sigma}$ is the honest-uniform strategy profile, then*

(i) $\mathbb{E}[\pi \mid \hat{\mathcal{M}}, \hat{\sigma}] = \mathbb{E}[\pi \mid \mathcal{M}, \sigma^*]$,

(ii) $\mathbb{E}[u_i \mid \hat{\mathcal{M}}, \hat{\sigma}, \theta] = \mathbb{E}[u_i \mid \mathcal{M}, \sigma^*, \theta]$, *and*

(iii) the strategy profile $\hat{\sigma}$ is an ex-post equilibrium of $\hat{\mathcal{M}}$.

Theorem 3.1 states that the honest-uniform profile is an ex-post equilibrium and provides the same payoffs to the agents and the designer as the original EPIC deterministic mechanism $\mathcal{M}$. Online Appendix C shows that this result also holds with Bayesian equilibrium as the solution concept.

The revelation principle states that each distribution of equilibrium outcomes can be generated by an EPIC direct mechanism where agents report truthfully. The analyst typically approaches mechanism design problems by focusing on EPIC direct mechanisms. Theorem 3.1 gives the analyst a route to do so, even if they are only interested in implementing deterministic mechanisms. The analyst can freely analyze the problem by using direct stochastic mechanisms, even if they are focused on a designer who chooses the optimal (potentially indirect) deterministic mechanism. The analyst can then think of a designer as implementing the latter mechanism.

4 Stochastic vs. Deterministic Direct Mechanisms

It is important to remark that Theorem 3.1 is not a deterministic version of the revelation principle. The mechanism constructed for the equivalence result is not direct, in the sense that it requires agents to both report a type and a number. This section explores the equivalence of stochastic and deterministic direct mechanisms under ex-post implementation.

4.1 Dominance

The example below shows that, when there are three or more outcomes, (i.e. $|Y| \geq 3$) deterministic direct mechanisms can be dominated by stochastic direct mechanisms, even when $n \geq 2$. So, a deterministic version of revelation principle for environments with three or more outcomes does not hold.[3]

Example 4.1. *This is a two agent version of Example 3.1, where only agent 1 has private information. Set $\Theta_1 = \{\underline{\theta}, \overline{\theta}\}$ and suppress reference to Θ_2. The outcome set is $Y = \{y_1, y_2, y_3\}$ and the payoff functions of the designer and agent 1 are the same as in Example 3.1. Thus, there is an EPIC direct mechanism that provides a payoff 1 to the designer. However, no deterministic direct mechanism can provide the designer with a guaranteed payoff of one. To see this, note that agent 2 does not have private information and so only agent 1 is active in a direct mechanism. So, as in Example 3.1, for each direct deterministic mechanism $\mathcal{M}$, agent 1 chooses an outcome from the set of reachable outcomes $Y(\mathcal{M})$. Since both types of agent 1 share the same strict preferences, they both select the same outcome.*

Notice that the non-equivalence holds in direct mechanisms since agent 2 is inactive. Theorem 3.1 shows that the equivalence can hold in a mechanism where both agents are active.

4.2 Equivalence

We now show that a deterministic version of a revelation principle holds in settings with binary outcomes and ex-post implementation. To do so, it will be useful to endow the set of mechanisms with a linear structure.

Fix a binary outcome set Y. Write $\mathrm{SM}(Y)$ for the set of finite signed measures over Y. It will be useful to consider the vector space

$$V := \left\{ m : \Theta \to \mathrm{SM}(Y) : m \text{ is measurable and } \sup_{(\theta,y)\in\Theta\times Y} m(\theta)(y) < \infty \right\},$$

[3] This is essentially the point in Strausz [2003]. In fact, he proves an even stronger statement. He shows that the set of direct deterministic mechanisms can be dominated not only by general stochastic mechanisms, but also by deterministic indirect mechanisms.

endowed with the operations of addition and multiplication by scalars inherited from SM(Y). Additionally, endow V with the supremum norm given by $||m||_\infty = \sup_{(\theta,y)\in\Theta\times Y} m(\theta)(y)$.

Write $M = \{m \in V \mid m(\theta) \in \Delta(Y) \text{ for each } \theta \in \Theta\}$. So, $m \in M$ if and only if (Θ, m) is a direct mechanism. Write $M^{\text{EPIC}} := \{m \in M \mid (\Theta, m) \text{ is EPIC}\}$ and write $M^{\text{D-EPIC}} := \{m \in M \mid (\Theta, m) \text{ is deterministic and EPIC }\}$. Notice that $M^{\text{D-EPIC}} \subsetneq M^{\text{EPIC}} \subseteq M \subsetneq V$ and that M^{EPIC} is bounded, closed, and convex. (See Lemma A.4.)

Lemma 4.1. *If $|Y| = 2$, then $M^{D\text{-}EPIC}$ is the set of extreme points of M^{EPIC}.*

Lemma 4.1 provides the key geometric property of the set of EPIC mechanisms. Notice that Example 4.1 shows that Lemma 4.1 does not hold in settings with three or more outcomes. In that example, if (Θ, m) is the optimal direct mechanism of the designer's problem, then m is an extreme point of M^{EPIC} but m is not deterministic.

Now we describe the designer's problem. Let $\Pi : V \to \mathbb{R}$ be a function given by

$$\Pi(m) = \int_{\theta\in\Theta} \int_{y\in Y} \pi(\theta, y)\ dm(\theta)\ d\mu.$$

Note that if $m \in M$, then $\Pi(m)$ is the designer's expected payoff of the direct mechanism (Θ, m), given the honest strategy profile. Moreover, Π is linear in V. We will compare the designer's problem of maximizing Π over the set M^{EPIC} to maximizing Π over $M^{\text{D-EPIC}}$. Say that the **designer's problem is well-behaved** if the set M^{EPIC} is compact and the mapping Π is continuous. So, each well-behaved designer's problem always has a solution.[4]

Theorem 4.1. *If $|Y| = 2$ and the designer's problem is well-behaved, then*

$$\max_{m\in M^{D\text{-}EPIC}} \Pi(m) = \max_{m\in M^{EPIC}} \Pi(m).$$

So, the designer's optimal mechanism is achieved by a direct deterministic mechanism.

Proof. Since M^{EPIC} is compact and Π is linear and continuous, Π achieves a maximum over M^{EPIC}. In addition, M^{EPIC} is a convex subset of a normed space (Lemma A.4) and $M^{\text{D-EPIC}}$ is the set of extreme points of M^{EPIC} (Lemma 4.1). Thus, the result follows by the Extreme Point Theorem (Ok [2011] p. 493). □

Theorem 4.1 is a deterministic version of the revelation principle for the case of binary outcomes. It implies that, as long as the designer's problem is well-behaved and the set of outcomes is binary, there is no loss in focusing on deterministic direct mechanisms.

There is a question of whether Theorem 4.1 extends to Bayesian incentive compatible mechanisms. This remains an open question. Importantly, the proof of Lemma 4.1 does not apply to Bayesian incentive compatible mechanisms.

[4]If $\Theta \times Y$ is finite then V has finite dimension so M^{EPIC} is a compact set and Π is continuous. In some other environments, like the provision of an indivisible good with one buyer, M^{EPIC} is compact and Π is continuous even though $\Theta \times Y$ is not finite. (See Börgers [2015].)

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Appendix A Additional Lemmata and Omitted Proofs

A.1 Proofs of Section 3

Lemma A.1. *Fix some Polish space Y and some $\nu \in \Delta(Y)$. There is a Borel measurable mapping $g : [0,1] \to Y$ such that, for each Borel $E \subseteq Y$, $\nu(E) = \lambda(g^{-1}(E))$.*

Proof. Fix some $X \subseteq [0,1]$ closed such that $|X| = |Y|$ and $0 \in X$. By the Borel Isomorphism Theorem, there is a bijective bimeasurable mapping $\phi : Y \to X$. (See Theorem 15.6 in Kechris (2012).) Let $\tilde{\varpi} \in \Delta(X)$ be the image measure of ν under ϕ and let $\varpi \in \Delta([0,1])$ be the extension of $\tilde{\varpi}$ to a measure on all of $[0,1]$.

Write $F : [0,1] \to [0,1]$ for the CDF associated with ϖ, i.e., a function with $F(x) = \varpi([0,x])$ for each $x \in [0,1]$. Observe that F is weakly increasing and so measurable. Moreover, F is continuous from the right. Call each z with $\{x \in [0,1] : F(x) < z\} \neq \emptyset$ non-trivial.

Define a new function $G : [0,1] \to [0,1]$ so that

$$G(z) = \begin{cases} \sup\{x \in X : F(x) < z\} & \text{if } z \text{ is non-trivial} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that G is a weakly increasing function and so measurable. Moreover, for each $x, z \in [0,1]$, $F(x) \geq z$ if and only if $G(z) \leq x$. To see this, first fix $x, z \in [0,1]$ so that $F(x) \geq z$. Since F is weakly increasing, for each $w \in [0,1]$ with $F(w) < z$, it must be that $w < x$; so, $G(z) \leq x$. Conversely, assume $F(x) < z$. Notice that $F(x) < z \leq 1$ implies $x < 1$. Since F is right-continuous and $x < 1$, there exists $\epsilon > 0$ so that $F(x+\epsilon) < z$. This implies $x + \epsilon \in \{w \in X : F(w) < z\}$ and so $G(z) \geq x + \epsilon > x$, as desired.

Now notice that, for each $x \in [0,1]$,

$$[0, F(x)] = \{z \in [0,1] : F(x) \geq z\} = \{z \in [0,1] : G(z) \leq x\} = G^{-1}([0,x]),$$

where the second equality follows from the fact that $F(x) \geq z$ if and only if $G(z) \leq x$ and the last equality follows from the fact that G is weakly increasing. So, for each $x \in [0,1]$

$$\varpi([0,x]) = F(x) = \lambda([0, F(x)]) = \lambda(G^{-1}([0,x])).$$

Since $\{[0,x] : x \in [0,1]\}$ generates the Borel sigma algebra on $[0,1]$, for each Borel measurable $A \subseteq [0,1]$, $\varpi(A) = \lambda(G^{-1}(A))$. (See Lemma 4.19 and Theorem 10.10 in Aliprantis and Border [2006] on the generating sigma algebra.)

Now notice that $G([0,1]) \subseteq X$. Clearly $G(z) \in X$ if $G(z) = 0$. So, suppose that $G(z) \neq 0$. Then z is non-trivial and, so, $\{x \in X : F(x) < z\}$ is a non-empty subset of X. Since X is closed, $\sup\{x \in X : F(x) < z\} \in X$.

Since $G([0,1]) \subseteq X$, write $G : [0,1] \to X$ for the restriction of G to the range X; this new function remains measurable. Now define $g : [0,1] \to Y$ so that $g = \phi^{-1} \circ G$ and note that g is measurable. Fix measurable $E \subseteq Y$. Notice that $g^{-1}(E) = G^{-1}(\phi(E))$. Thus

$$\nu(E) = \varpi(\phi(E)) = \lambda(G^{-1}(\phi(E))) = \lambda(g^{-1}(E)),$$

where the first equality follows from the fact that ϕ is a bijective and bimesurable. □

Lemma A.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a random variable $\mathbf{u}$ that follows a uniform distribution on $[0,1]$. Then, for each $c \in \mathbb{R}$, $\text{res}(c+\mathbf{u})$ follows uniform distribution on $[0,1]$.*

Proof. Fix $c \in \mathbb{R}$ and note that there exists an integer x so that $c = x + \text{res}(c)$. It suffices to show that $\mathbb{P}(\text{res}(c+\mathbf{u}) \leq a) = a$.

First suppose $\text{res}(c) \leq a$. Fix a realization of the random variable $\mathbf{u}$, namely u. If $c+u < x+1$, then $\text{res}(c+u) \leq a$ if and only if $u \leq a - \text{res}(c)$. If $c+u \geq x+1$, then $\text{res}(c+u) \leq a$ if and only if $u > 1-\text{res}(c)$. (This uses the fact that $\text{res}(\text{res}(c)+u) \leq a$, since $\text{res}(c) \leq a$ implies $\text{res}(c)+u \leq 1+a$.) Thus,

$$\mathbb{P}(\text{res}(c+\mathbf{u}) \leq a) = \mathbb{P}(0 \leq \mathbf{u} < a - \text{res}(c)) + \mathbb{P}(1-\text{res}(c) \leq \mathbf{u} \leq 1) = a - \text{res}(c) + \text{res}(c) = a.$$

Now suppose $\text{res}(c) > a$. Fix a realization of the random variable $\mathbf{u}$, namely u. Notice $\text{res}(c+u) \leq a$ if and only if $u \in [1-\text{res}(c), 1-\text{res}(c)+a]$. (This uses the fact that $\text{res}(\text{res}(c)+u) \leq a$ if $\text{res}(c)+u \geq 1$ and $\text{res}(c)+u \leq 1+a$.) Thus,

$$\mathbb{P}(\text{res}(c+\mathbf{u}) \leq a) = \mathbb{P}(1-\text{res}(c) \leq \mathbf{u} \leq 1-\text{res}(c)+a) = a.$$

So $\mathbb{P}(\text{res}(c+\mathbf{u}) \leq a) = a$ as desired. □

Lemma A.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with n independent uniform random variables $\mathbf{u}_1, \ldots \mathbf{u}_n$ on $[0,1]$. Then, for each $c \in \mathbb{R}$, $\text{res}\left(c + \sum_{k=1}^n \mathbf{u}_k\right)$ follows a uniform distribution on $[0,1]$.*

Proof. Fix $c \in \mathbb{R}$. If $n = 1$, Lemma A.2 establishes the result. Assume that $n > 1$. Let $\mathbb{P}(\,\cdot \mid \sum_{k=1}^{n-1} \mathbf{u}_k)$ be a regular version of conditional probability for the sigma-algebra generated by $\sum_{k=1}^{n-1} \mathbf{u}_k$. Note that for each $a \in (0,1)$ and each $c' \in \mathbb{R}$,

$$\mathbb{P}\left(\text{res}\left(c + \textstyle\sum_{k=1}^n \mathbf{u}_k\right) \leq a \;\middle|\; \textstyle\sum_{k=1}^{n-1} \mathbf{u}_k = c'\right) = \mathbb{P}\left(\text{res}(c + c' + \mathbf{u}_n) \leq a \;\middle|\; \textstyle\sum_{k=1}^{n-1} \mathbf{u}_k = c'\right).$$

In addition, by independence, it follows that

$$\mathbb{P}\left(\text{res}(c + c' + \mathbf{u}_n) \leq a \;\middle|\; \textstyle\sum_{k=1}^{n-1} \mathbf{u}_k = c'\right) = \mathbb{P}\left(\text{res}(c + c' + \mathbf{u}_n) \leq a\right), \qquad \text{a.s.}$$

Note that $\mathbb{P}\left(\text{res}(c + c' + \mathbf{u}_n) \leq a\right) = a$. (See Lemma A.3.) Hence, for each $c' \in \mathbb{R}$,

$$\mathbb{P}\left(\text{res}\left(c + \textstyle\sum_{k=1}^n \mathbf{u}_k\right) \leq a \;\middle|\; \textstyle\sum_{k=1}^{n-1} \mathbf{u}_k = c'\right) = a, \qquad \text{a.s.}$$

So, by integrating over $c' \in \mathbb{R}$, $\mathbb{P}\left(\text{res}\left(c + \sum_{k=1}^{n} \mathbf{u}_k\right) \leq a\right) = a$, as desired. □

Proof of Lemma 3.1. Begin with (i). Under the uniform-honest strategy profile $\hat{\sigma}$, the supplemental reports $s = (s_1, ..., s_n)$ are independent and uniformly distributed on $[0,1]$. By Lemma A.3, $\text{res}\left(\sum_{0=1}^{n} s_i\right)$ follows a uniform distribution $[0,1]$. Write λ^n for the Lebesgue measure in $[0,1]^n$ and Fix $\theta \in \Theta$. Since $s \in [0,1]^n$ is uniformly distributed, $\hat{y}(\theta, s) = g_\theta\left(\text{res}\left(\sum_{i=1}^{n} s_i\right)\right)$ follows the same distribution as $m(\theta)$. (See Lemma A.1.) Thus,

$$\int_{s\in[0,1]^n} \pi(\theta, \hat{y}(\theta, s))\ d\lambda^n = \int_{y\in Y} \pi(\theta, y)\ dm(\theta).$$

Therefore,

$$\begin{aligned}
\mathbb{E}[\pi \mid \hat{\mathcal{M}}, \hat{\sigma}] &= \int_{\theta\in\Theta} \int_{s\in[0,1]^n} \pi(\theta, \hat{y}(\theta, s))\ d\lambda^n\ d\mu \\
&= \int_{\theta\in\Theta} \int_{y\in Y} \pi(\theta, y)\ dm(\theta)\ d\mu \\
&= \mathbb{E}[\pi \mid \mathcal{M}, \sigma^*].
\end{aligned}$$

We show (ii). Under the uniform-honest strategy profile $\hat{\sigma}$, $s_1, .., s_n$ are independent and uniformly distributed on $[0,1]$. Then, for each fixed $s'_i \in [0,1]$, $\text{res}\left(s'_i + \sum_{k=1, i\neq k}^{n} s_k\right)$ follows a uniform distribution on $[0,1]$. (Use Lemma A.3 with $c = s'_i$ and $\mathbf{u}_k = s_j$ for $j \neq i$). Write λ^{n-1} for the Lebesgue measure on $[0,1]^{n-1}$. Then $\hat{y}((\theta'_i, \theta_{-i}), (s'_i, s_{-i})) = g_{(\theta'_i, \theta_{-i})}(\text{res}(s'_i + \sum_{j\neq i} s_j)))$ follows the same distribution as $m(\theta'_i, \theta_{-i})$ and

$$\int_{s_{-i}\in[0,1]^{n-1}} u_i(\theta, \hat{y}((\theta'_i, \theta_{-i}), (s'_i, s_{-i})))\ d\lambda^{n-1} = \int_{y\in Y} u_i(\theta, y)\ dm(\theta'_i, \theta_{-i}).$$

Therefore,

$$\begin{aligned}
\mathbb{E}[u_i \mid \hat{\mathcal{M}}, ((\theta'_i, s'_i), \hat{\sigma}_{-i}), \theta] &= \int_{s_{-i}\in[0,1]^{n-1}} u_i(\theta, \hat{y}((\theta'_i, \theta_{-i}), (s'_i, s_{-i})))\ d\lambda^{n-1} \\
&= \int_{y\in Y} u_i(\theta, y)\ dm(\theta'_i, \theta_{-i}) \\
&= \mathbb{E}[u_i \mid \mathcal{M}, (\theta'_i, \sigma^*_{-i}), \theta],
\end{aligned}$$

as desired. □

A.2 Proofs of Section 4

Lemma A.4. *The set M^{EPIC} is non-empty, bounded, closed, and convex.*

Proof. First we show that M^{EPIC} is not empty. Let $m : \Theta \to \Delta(Y)$ be a mapping such that $m(\theta) = m(\theta')$ for each $\theta, \theta' \in \Theta$. Since m does not depend of the agents' reports, (Θ, m) is an EPIC direct mechanism. So, $m \in M^{\text{EPIC}} \neq \emptyset$. Also note that M^{EPIC} is bounded, since $||m||_\infty \leq 1$ for each mechanism.

It remains to show that M^{EPIC} is closed and convex. Fix θ_i, θ_i' and θ_{-i}. Write $W_i(\theta_i, \theta_i', \theta_{-i})$ for the set of vectors $m \in M$ that satisfies the EPIC constraint

$$\int_{y \in Y} (u_i(\theta_i, \theta_{-i}, y) - u_i(\theta_i', \theta_{-i}, y)) \; dm(\theta_i, \theta_{-i}) \geq 0 \tag{1}$$

for $(\theta_i, \theta_i', \theta_{-i})$. Notice that the constraint is linear and has a weak inequality. Hence, the set $W_i(\theta_i, \theta_i', \theta_{-i})$ is closed and convex. Since

$$M^{\text{EPIC}} = \bigcap_{i \in I} \bigcap_{\theta_i \in \Theta_i} \bigcap_{\theta_i' \in \Theta_i} \bigcap_{\theta_{-i} \in \Theta_{-i}} W_i(\theta_i, \theta_i', \theta_{-i}),$$

it follows that M^{EPIC} is convex and closed. □

Say $m' \in M$ is a **uniform monotone transformation** of $m \in M$ if, for each $y \in Y$, there is a weakly increasing function $f_y : [0,1] \to [0,1]$ such that $m'(\theta)(y) = f_y(m(\theta)(y))$ for each $\theta \in \Theta$.

Lemma A.5. *Assume that $|Y| = 2$. If $m \in M^{EPIC}$, and $m' \in M$ is a uniform monotone transformation of m, then $m' \in M^{EPIC}$.*

Proof. Let $Y = \{\underline{y}, \overline{y}\}$ and suppose that m' is a uniform monotone transformation of $m \in M^{\text{EPIC}}$. Then, for each $y \in Y$, there exists a weakly increasing function f_y such that $m'(\theta)(y) = f_y(m(\theta)(y))$. Fix $\theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$. Since m is ex-post implementable, then for each $\theta_{-i} \in \Theta_{-i}$,

$$\sum_{y \in Y} u_i(\theta_i, \theta_{-i}, y) \cdot m(\theta_i, \theta_{-i})(y) \geq \sum_{y \in Y} u_i(\theta_i, \theta_{-i}, y) \cdot m(\theta_i', \theta_{-i})(y). \tag{2}$$

It suffices to show

$$\sum_{y \in Y} u_i(\theta_i, \theta_{-i}, y) \cdot m'(\theta_i, \theta_{-i})(y) \geq \sum_{y \in Y} u_i(\theta_i, \theta_{-i}, y) \cdot m'(\theta_i', \theta_{-i})(y). \tag{3}$$

If $u_i(\theta_i, \theta_{-i}, \underline{y}) = u_i(\theta_i, \theta_{-i}, \overline{y})$, Equation (3) is trivially satisfied. So, without loss of generality suppose $u_i(\theta_i, \theta_{-i}, \underline{y}) > u_i(\theta_i, \theta_{-i}, \overline{y})$. (If $u_i(\theta_i, \theta_{-i}, \underline{y}) < u_i(\theta_i, \theta_{-i}, \overline{y})$, an analogous argument applies.)

In this case, Equation (2) implies that $m(\theta_i, \theta_{-i})(\underline{y}) \geq m(\theta_i', \theta_{-i})(\underline{y})$. Since $f_{\underline{y}}$ is weakly increasing, $f_{\underline{y}}(m(\theta_i, \theta_{-i})(\underline{y})) \geq f_{\underline{y}}(m(\theta_i', \theta_{-i})(\underline{y}))$. So, $m'(\theta_i, \theta_{-i})(\underline{y}) \geq m'(\theta_i', \theta_{-i})(\underline{y})$. From this, Equation (3) follows. □

***Proof of Lemma** 4.1.* Let $Y = \{\overline{y}, \underline{y}\}$. Observe that any $m \in \mathcal{M}^{\text{D-EPIC}}$ is an extreme point of M^{EPIC}. We show that, if $m \in M^{\text{EPIC}}$ is not deterministic, then m is not an extreme point of M^{EPIC}. To do so, we construct $m_1, m_2 \in M^{\text{EPIC}}$ so that $m = \frac{1}{2}m_1 + \frac{1}{2}m_2$ and $m_1 \neq m \neq m_2$.

To construct m_1 and m_2, Let $f_1 : [0,1] \to [0,1]$ and $f_2 : [0,1] \to [0,1]$ be two functions defined by

$$f_1(\alpha) = \begin{cases} 0 & \text{if } \alpha \leq \frac{1}{2} \\ 2\alpha - 1 & \text{if } \alpha > \frac{1}{2} \end{cases} \qquad \text{and} \qquad f_2(\alpha) = \begin{cases} 2\alpha & \text{if } \alpha \leq \frac{1}{2} \\ 1 & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

Notice that f_1 and f_2 are weakly increasing and satisfy the following:

$$f_1(\alpha) < \alpha < f_2(\alpha) \qquad \text{for each } \alpha \in (0,1), \text{ and} \tag{4}$$

$$\frac{1}{2}\, f_1(\alpha) + \frac{1}{2}\, f_2(\alpha) = \alpha \qquad \text{for each } \alpha \in [0,1]. \tag{5}$$

Define m_1 and m_2 in the following way:

$$m_k(\theta)(\overline{y}) = f_k(m(\theta)(\overline{y})), \quad m_k(\theta)(\underline{y}) = 1 - m_k(\theta)(\overline{y}) \qquad \text{for all } \theta \in \Theta, k \in \{1,2\}.$$

Note that by, Equation (5), that $\frac{1}{2}m_1 + \frac{1}{2}m_2 = m$. Secondly, since m is not deterministic, there is a $\theta \in \Theta$ such that $m(\theta)(\overline{y}) \in (0,1)$. Then, by Equation (4), $m_1 \neq m \neq m_2$. It suffices to show that $m_1, m_2 \in M^{\text{EPIC}}$. Since f_1 and f_2 are both increasing, it follows that m_1 m_2 are both uniform monotone transformations of m. By Lemma A.5, the mechanisms m_1 and m_2 are EPIC. □

Deterministic Mechanism Design

Online Appendix: Not Intended for Publication

Ernesto Rivera Mora

Appendix B Dominance of Stochastic Mechanisms

This section shows that this the designer is strictly better off by using a randomization device in a broad class of settings with one agent and three or more outcomes.

Fix an environment. In this environment, say that **stochastic mechanisms dominate deterministic mechanisms** if there exist a stochastic mechanism and an ex-post equilibrium thereof that gives the designer strictly higher payoffs than any ex-post equilibrium of any deterministic mechanism.

Proposition B.1. *Let $n = 1$. Suppose there is some $\{y_1, y_2, y_3\} \subseteq Y$ and some $\widetilde{\Theta} \subseteq \Theta$ so that the following hold:*

(i) For each $\theta \in \widetilde{\Theta}$, $u(\theta, y_1) < u(\theta, y_2) < u(\theta, y_3)$.

(ii) There is an $x \in (0, 1)$ and disjoint sets $\widetilde{\Theta}_1, \widetilde{\Theta}_2 \subseteq \widetilde{\Theta}$ with $\mu(\widetilde{\Theta}_1) > 0$ and $\mu(\widetilde{\Theta}_2) > 0$ so that

- *(a) $\theta \in \widetilde{\Theta}_1$ implies $u(\theta, y_2) \leq xu(\theta, y_1) + (1 - x)u(\theta, y_3)$, and*
- *(b) $\theta \in \widetilde{\Theta}_2$ implies $u(\theta, y_2) \geq xu(\theta, y_1) + (1 - x)u(\theta, y_3)$.*

Then there is an objective of the designer $\pi : \Theta \times Y \to \mathbb{R}$ so that stochastic mechanisms dominate deterministic mechanisms.

To better understand the assumptions of Proposition B.1, consider the case where Θ is countable. In that case, if $|\widetilde{\Theta}| \geq 2$, then condition (i) implies condition (ii). So, there, the proposition simply requires that there are two types that each prefer y_3 to y_2 to y_1.

Proof. We will construct some objective $\pi : \Theta \times Y \to \mathbb{R}$ so that the following hold:

(1) There is a (stochastic) EPIC direct mechanism that gives the designer a expected payoff of 1 under the honest strategy profile.

(2) For each deterministic mechanism, and each ex-post equilibrium thereof, the designer's expected payoff is strictly less than 1.

To do so, we make use of the sets $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ in the statement of the proposition. We use these sets to construct a measurable partition $\{\overline{\Theta}_1, \overline{\Theta}_2\}$ of Θ as follows. First, set

$$\overline{\Theta}_1 = \{\theta \in \Theta : u(\theta, y_2) \leq x \cdot u(\theta, y_1) + (1 - x) \cdot u(\theta, y_3)\} \setminus \widetilde{\Theta}_2.$$

Then, set $\overline{\Theta}_2 = \Theta \setminus \overline{\Theta}_1$. To see that $\overline{\Theta}_1$ and $\overline{\Theta}_2$ are measurable, it suffices to show that the set $\{\theta \in \Theta : u(\theta, y_2) \leq x \cdot u(\theta, y_1) + (1 - x) \cdot u(\theta, y_3)\}$ is measurable. But this follows from the fact that $u(\cdot, y_2) - xu(\cdot, y_1) - (1 - x)u(\cdot, y_3)$ is continuous.

Now construct the objective $\pi : \Theta \times Y \to \mathbb{R}$. Set $\pi(\theta, y) = 1$ if $(\theta, y) \in \overline{\Theta}_1 \times \{y_1, y_3\}$ or $(\theta, y) \in \overline{\Theta}_2 \times \{y_2\}$. Otherwise, set $\pi(\theta, y) = 0$.

First we show (1). Write $\alpha_1 \in \Delta(Y)$ for the measure such that $\alpha_1(y_1) = x$ and $\alpha_1(y_3) = 1 - x$. Write $\alpha_2 \in \Delta(Y)$ for the measure that selects y_2 with probability 1. Define the direct mechanism (Θ, m) so that $m(\theta) = \alpha_1$ if $\theta \in \overline{\Theta}_1$ and $m(\theta) = \alpha_2$ if $\theta \in \overline{\Theta}_2$. Under the honest strategy, each type of the agent gets her favorite lottery among α_1 and α_2. (This uses the definition of $\overline{\Theta}_1$ and $\overline{\Theta}_2$.) Hence, the mechanism (Θ, m) is EPIC. Moreover, (Θ, m) provides to the designer an expected payoff of 1 under the honest strategy profile.

Now we show (2). Fix a deterministic mechanism $\mathcal{M}' = (R, m)$. For each $r \in R$, $\text{Supp}\,(m(r)) \subsetneq Y$ is a singleton. Write $R_k = \{r \in R : \text{Supp}\,(m(r)) = y_k\}$ and fix an ex-post equilibrium σ. We will show that the designer gets an expected payoff strictly less than 1 under σ.

To show this, first suppose that $R_2 = \emptyset$. Since $\mu(\overline{\Theta}_2) \geq \mu(\widetilde{\Theta}_2) > 0$, the designer's expected payoff is strictly less than 1, for any strategy the agent chooses.

So, suppose that $R_2 \neq \emptyset$ but $R_3 = \emptyset$. In that case, $R_1 = R_1 \cup R_3$. Since $R_2 \neq \emptyset$, for each $\theta \in \widetilde{\Theta}_1$, $\sigma(\theta)(R_1 \cup R_3) = 0$. (Otherwise, the agent would have an incentive to deviate, at least, to reports in R_2.) Since $\mu(\widetilde{\Theta}_1) > 0$, the designer's expected payoff is strictly less than 1, under σ.

Finally, suppose that $R_2, R_3 \neq \emptyset$. In that case, $\theta \in \widetilde{\Theta}_2$, $\sigma(\theta)(R_2) = 0$. (Otherwise, the agent would have an incentive to deviate to deviate, at least, to reports in R_3.) Since $\mu(\widetilde{\Theta}_2) > 0$, the designer's expected payoff is strictly less than 1, under σ. $\square$

Appendix C Bayesian Incentive Compatibility

This Appendix shows an analogue of Theorem 3.1 for Bayesian equilibrium.

Note, there is a function $\mu_i : \Theta_i \to \Delta(\Theta_{-i})$, where $\mu_i(\theta_i)$ captures the conditional probability of θ_{-i} given θ_i. Write

$$\mathbb{E}[u_i \mid \mathcal{M}, \sigma, \theta_i] = \int_{\Theta_{-i}} \int_R \int_Y u_i(\theta_i, \theta_{-i}, y)\ dm(r)\ d\sigma(\theta_i, \theta_{-i})\ d\mu_i(\theta_i).$$

for type θ_i's expected utility, under strategy profile σ in the mechanism $\mathcal{M}$. Likewise write $\mathbb{E}[u_i \mid \mathcal{M}, (r_i, \sigma_{-i}), \theta_i]$ for type θ_i's expected utility given the report r_i and the strategy profile σ_{-i}, in the mechanism $\mathcal{M}$.

Definition C.1. *Fix a mechanism $\mathcal{M} = (R, m)$ and a strategy profile σ. Call σ a **Bayesian equilibrium** of $\mathcal{M}$ if, for each $\theta_i \in \Theta_i$ and each $r_i \in R_i$, $\mathbb{E}[u_i \mid \mathcal{M}, \sigma, \theta_i] \geq \mathbb{E}[u_i \mid \mathcal{M}, (r_i, \sigma_{-i}), \theta_i]$.*

Fix a direct mechanism $\mathcal{M}$. Say $\mathcal{M}$ satisfies **Bayesian incentive compatibility (BIC)** if the honest strategy profile is a Bayesian equilibrium.

Theorem C.1. *Let $n \geq 2$. Fix a a BIC direct mechanism $\mathcal{M}$ and let σ^* be the honest strategy profile thereof. If $\hat{\mathcal{M}}$ is a deterministic mechanism induced by $\mathcal{M}$ and $\hat{\sigma}$ is the uniform-honest strategy profile then*

(i) $\mathbb{E}[\pi \mid \hat{\mathcal{M}}, \hat{\sigma}] = \mathbb{E}[\pi \mid \mathcal{M}, \sigma^*]$,

(ii) $\mathbb{E}[u_i \mid \hat{\mathcal{M}}, \hat{\sigma}, \theta_i] = \mathbb{E}[u_i \mid \mathcal{M}, \sigma^*, \theta_i]$, *and*

(iii) The strategy profile $\hat{\sigma}$ is a Bayesian equilibrium of $\hat{\mathcal{M}}$.

Proof. Fix a BIC direct mechanism $\mathcal{M}$ and let $\hat{\mathcal{M}}$ be the deterministic mechanism it induces. Let $\hat{\sigma}$ the the honest-uniform strategy profile of $\hat{\mathcal{M}}$. Lemma 3.1 states (i) and, moreover, that, for each $(\theta_i', s_i') \in R_i$ and $\theta \in \Theta$, $\mathbb{E}[u_i \mid \hat{\mathcal{M}}, ((\theta_i', s_i'), \hat{\sigma}_{-i}), \theta] = \mathbb{E}[u_i \mid \mathcal{M}, (\theta_i', \sigma_{-i}^*), \theta]$. By taking the expectation over all profiles $\theta \in \Theta$ with a given type θ_i, it follows that $\mathbb{E}[u_i \mid \hat{\mathcal{M}}, ((\theta_i', s_i'), \hat{\sigma}_{-i}), \theta_i] = \mathbb{E}[u_i \mid \mathcal{M}, (\theta_i', \sigma_{-i}^*), \theta_i]$, and thus (ii) holds. Finally notice that

$$\begin{aligned}\mathbb{E}[u_i \mid \hat{\mathcal{M}}, ((\theta_i, s_i'), \hat{\sigma}_{-i}), \theta_i] &= \mathbb{E}[u_i \mid \mathcal{M}, (\theta_i, \sigma_{-i}^*), \theta_i] \\ &\geq \mathbb{E}[u_i \mid \mathcal{M}, (\theta_i', \sigma_{-i}^*), \theta_i] = \mathbb{E}[u_i \mid \hat{\mathcal{M}}, ((\theta_i', s_i'), \hat{\sigma}_{-i}), \theta_i].\end{aligned}$$

where the inequality follows from BIC of $\mathcal{M}$. Thus, the honest-uniform strategy profile is a Bayesian equilibrium. □