Information Markets in Games*

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Abstract

Markets for information are ubiquitous in modern society. Understanding how information providers choose to sell information is important for designing policies that improve efficiency. This paper studies markets for information when two uninformed agents play a quadratic game and characterizes the revenue maximizing information-selling schemes. The optimal way to sell information depends on the degree to which agents' actions are strategic substitutes or complements. In the case of strategic complements, it is always optimal to sell perfect information to both agents. However, in the case of strategic substitutes, there is a trade-off; revealing more information increases the correlation between the agents' actions, which in turn decreases the value of information. If the degree of strategic substitutability is sufficiently high, it is optimal for the seller to obfuscate information. Depending on the degree of substitutability, it is either optimal to sell perfect information to exactly one agent, or to sell a noisy signal to both.

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1 Introduction

This paper focuses on two important properties of information: 1) information is replicable, so can be sold to multiple buyers; 2) The value of information to one buyer depends on whether other buyers acquired the information (or not). This paper investigates the implications of these two properties for the behavior of information sellers and buyers.

As an example, consider the firms Airbus and Boeing as information buyers, and a consulting firm as an information seller. Here, Airbus and Boeing play a competitive game where their utilities depend on their production/pricing actions and the demand level θ. The parameter θ is unknown. However, the consultant can design a study that reveals θ with the goal of selling this information to Airbus and Boeing. Nonetheless, competing against an informed firm is different from competing against an uninformed one. Thus, the value that Airbus assigns to observing θ may not only depend on the information itself, but also on whether Boeing has access to θ or not. One possibility is that Airbus' valuation of information is independent of whether Boeing also has access to θ. A second possibility is that Airbus' valuation decreases if Boeing also has access to θ, creating a negative externality. A third possibility is that Airbus' valuation increases if Boeing also has access to θ, creating a positive externality.

The informational externalities between the buyers are relevant because they impact the optimal way a seller offers information. For instance, the consultant could offer to reveal θ to both firms or to only one firm. The key observation is that, as long as informational externalities are sufficiently negative, then offering information to both firms may not maximize the consultant's revenue. Instead, the consultant may increase her revenue by revealing θ only to Airbus and signing a non-disclosure contract that promises to not reveal θ to Boeing. Thus, if Airbus alone had access to θ, it would pay more for this deal than what both firms would pay if they both had access to θ.

The first part of the paper explores the strategic interaction between a monopolistic information seller and two information buyers that play a game. In this game the payoffs depend on an unknown parameter θ and the actions of both players. There are two stages. In stage one, the seller chooses a selling scheme. The selling scheme specifies the buyers' participation fees and the information structure, i.e. the rules of what the seller reveals about θ to each buyer. Buyers observe the selling scheme and decide to participate and pay the fee, or to pass. In stage two, both buyers play the competitive game of incomplete information where they use the information acquired from the seller. The seller is indifferent about the outcome of the game, and only wishes to maximize the revenue obtained by selling information to the buyers.

To make the problem tractable but sufficiently general, I focus on the case where the game's payoffs take a quadratic and symmetric form. This family of games includes many relevant economic environments, like some price and quantity competition oligopoly models where the unknown parameter θ represents a scalar of the demand level. In addition, the model assumes that: (1) the unknown parameter θ is drawn from a common prior, (2) the seller can privately communicate to each buyer, (3) the seller commits to the selling scheme she offers.

The first result of this paper characterizes behavior in the Bayesian game induced by any information structure the seller chooses to offer. It states there is a unique Bayesian equilibrium in which the agents' strategies and expected payoffs are expressed by a simple and convergent infinite sum of the agents' hierarchies of expectations about the state θ. That is, i's action and expected payoff depend not only on the information i gets about θ, but also on the information i gets about the information $-i$ gets about θ, the information i gets about the information $-i$ gets about the information i gets about θ, and so on. As a second result, this paper characterizes the value that each information buyer assigns to each information structure, i.e. the value each buyer is willing to pay to receive the private message associated with the information structure. The valuation of player i of an information structure is the variance of i's action that the information structure induces. That is, the greater the degree to which i's action moves according to the message i receives, the more i is willing to pay for it.

The characterization of value of information structures provides the main contribution of the paper. It provides a characterization of the selling schemes that maximize the seller revenue and shows that they depend on the degree to which agents' actions are strategic substitutes or complements. For instance, provided that actions are strategic complements, revealing the state to both players maximizes the seller's revenue; it is sub-optimal to obfuscate information to either player. However, if actions are strategic substitutes, this may not hold. With strategic substitutes there is a trade-off between the gains from providing more information and the losses arising from increased correlation between the agents' actions. Different selling schemes may be optimal depending on the degree of the strategic effects. First, with sufficiently low strategic effects, revealing the state to both players is optimal. In this case players care more about the information itself in comparison on what the other agent will do. Second, with stronger strategic effects, obfuscating information is optimal and can take two different forms. Depending on the degree of strategic substitutability, the seller may opt to completely reveal the state to only one player and reveal nothing to the other player, or provide noisy signals to both agents.

In addition, the paper provides an analysis of revenue maximization when the seller is constrained to publicly communicate with the players and shows that obfuscation is never optimal. This implies that if actions are strategic substitutes with strong strategic effects, then the seller gets strictly better off by privately communicating with the players.

1.1 Literature Review

This paper transforms the sellers' optimization problem into an information design problem in the spirit of Kamenica and Gentzkow (2011) (see also Rayo and Segal (2010)). Similarly to Mathevet et al. (2020), the buyers play a game so the seller's objective depends on all the hierarchies of beliefs. However, here the seller's objective depends only on the players' hierarchies of expectations of the unknown parameter, and not on the full description of hierarchies of beliefs.

The seminal work of Admati and Pfleiderer (1986) is the first to study optimal information seller schemes of a monopolistic seller. They analyze a framework where a monopolistic information seller proves information to a continuum of agents rewarding the value of an asset. They show that it is always optimal to sell private and noisy signals to the information buyers. This helps the seller to protect the value of information by decreasing the amount of information that *leaks* trough the asset price. Similarly, this paper shows that obfuscating information may be the optimal way to sell information.

The paper is related to work by Bergemann and Morris (2016). Using the notion of Bayes correlated equilibrium (BCE), Bergemann provides a characterization of the distribution of equilibrium outcomes in the Bayesian games generated by all possible information structures. However, it is silent about which distribution of outcomes corresponds to which information structures. Bergemann and Morris (2013) characterizes the set of normally distributed BCE in quadratic games. In contrast, this paper does not impose distributional assumptions and characterizes Bayesian equilibrium outcomes and players' valuation for each information structure in quadratic games.

Finally, this paper is related to earlier work on strategic information sharing in oligopoly models. Gal-Or (1985) shows that competitors never share private information about the demand intercept in Cournot models. More recent work in Goltsman and Pavlov (2014) investigates conditions under which firms can communicate their costs using a mediation protocol. These papers explore to what extent firms can trade information for information. In contrast, this paper explores up to what extent firms can trade information for money.

2 Model

Throughout the paper, take the following conventions. Endow a compact metric space C with its Borel sigma algebra. Denote the set of Borel probability measures by $\Delta(C)$. Endow the product of topological spaces with the product topology and endow $\Delta(C)$ with the topology of weak convergence. All omitted proofs in the main text are in the Appendix.

2.1 Environment

The set of players is $I = \{1,2\}$. There is a compact state space $\Theta \subseteq [0,\infty)$. The state θ is drawn from a common prior $\mu \in \Delta(\Theta)$ with full support The realized state is unknown to the players. Each player i chooses an action in a set A_i. The paper covers two cases: $A_1 = A_2 = [0,\infty)$ and $A_1 = A_2 = \mathbb{R}$.

Player i's utility function $u_i : \Theta \times A_i \times A_{-i} \to \mathbb{R}$ takes a quadratic form

$$u_i(\theta, a_i, a_{-i}) = \theta a_i - a_i^2 - \lambda a_i a_{-i},$$

where λ is a commonly known parameter. The action a_i impacts i's utility in three ways. First, there is a benefit of choosing a_i; that benefit is increasing in the state θ. Second, a_i has a quadratic cost a_i^2. Third, a_i has a strategic effect given by the term $-\lambda a_i a_{-i}$. If $\lambda > 0$, then the actions are **strategic substitutes**, i.e. the higher the action of the co-player, the greater the incentive to decrease one's own action. If $\lambda < 0$, then actions are **strategic complements**, i.e. the higher the action of the co-player, the greater the incentive to increase one's own action. If $\lambda = 0$, the co-player's action does not affect one's own utility, so both players face an individual decision problem.

The paper focuses on the case of mild strategic effects, i.e. environments where $|\lambda|$ is not large.[1] It is assumed that $\lambda \in (-2, \overline{\lambda})$. The model considers two cases. In the first case, $A_1 = A_2 = \mathbb{R}$ and $\overline{\lambda} = 2$. In the second case case, $A_1 = A_2 = [0,\infty)$ and $\overline{\lambda} = \frac{2\min(\Theta)}{\max(\Theta)}$. Notice that, if the domain of uncertainty is sufficiently small, i.e. if $\min(\Theta)$ is close to $\max(\Theta)$, then this latter bound is close to 2.

The following are two examples of relevant economic interactions in which the players' utilities have this quadratic form.

Example 2.1. Price Competition. Players are oligopolies engaging in price competition with differentiated products. Each firm i chooses a price $a_i \in A_i = [0,\infty)$. The demand faced by firm i is given by $Q_i = x - ba_i + ga_{-i}$, so profits are $\Pi_i = xa_i - ba_i^2 + ga_i a_{-i}$, where where $b, g, x > 0$. Firms know all parameters except the demand intercept x, which is drawn from $[\underline{x}, \overline{x}] \subset [0,\infty)$. Firm i's utility can be written as $u_i(\theta, a_i, a_{-i}) = \theta a_i - a_i^2 - \lambda a_i a_{-i}$, where $\theta = \frac{x}{b}$ and $\lambda = -\frac{g}{b}$. Note that λ is negative, so actions are strategic complements. The assumption that $\lambda > -2$ holds provided that $g < 2b$. That is, if i's demand reacts more to i's price than to $-i$'s price.

Example 2.2. Quantity Competition. Players are oligopolies engaging in quantity competition with differentiated products. Each firm i chooses a quantity $a_i \in A_i = [0,\infty)$ to produce. The inverse demand faced by firm i is given by $P_i = x - ba_i - ga_{-i}$, its costs are given by $C_i = da_i + ea_i^2$, so its profits are $\Pi_i = (x-d)a_i - (b+e)a_i^2 - ga_i a_{-i}$, where $x > d$, $b+e > 0$

[1]Appendix C provides an analysis for cases with extreme strategic effects (when $|\lambda| \geq 2$).

and $g > 0$. Firms know all parameters except the demand intercept x, which is drawn from $[\underline{x}, \overline{x}] \subset [d, \infty)$. Firm i's utility can be written as $u_i(\theta, a_i, a_{-i}) = \theta a_i - a_i^2 - \lambda a_i a_{-i}$, where $\theta = \frac{x-d}{b+e}$ is the unknown parameter and $\lambda = \frac{g}{b+e}$. Note that λ is positive, so actions are strategic substitutes. The assumption $\lambda < \overline{\lambda}$ holds provided that $\underline{x} - \frac{g}{2(b+e)}\overline{x} > d(1 - \frac{g}{2(b+e)})$ i.e. if the range of uncertainty of x is not to big.

Through the paper it will be convenient to parametrize the game by the parameter λ. Write $G(\lambda)$ by the game parametrized by λ.

2.2 Information-Selling Schemes

There is a monopolistic information seller who can provide signals about the state θ. Information about θ could lead the players to make better decisions, and they may thus be willing to pay to observe messages about θ. The seller's goal is to maximize revenue by selling messages to the players.

Call the tuple $\mathcal{I} = (M, \pi)$ an **information structure**, where $M := M_1 \times M_2$ is a compact metric space of messages profiles and $\pi : \Theta \to \Delta(M)$ is a message protocol, so that the mapping $\pi(\cdot)(E) : \Theta \to [0, 1]$ is measurable for each measurable set $E \subseteq M$. A selling scheme consists of $(\mathcal{I}, p_1, p_2)$, where $p_i \geq 0$ is a participation price for player i. All information structures are costless for the seller. The objective of the seller is to find a selling scheme that maximizes revenue.

The timing is given as follows: First, the seller chooses and commits to a selling scheme $(\mathcal{I}, p_1, p_2)$. Second, players observe $(\mathcal{I}, p_1, p_2)$ and simultaneously decide to participate or not. If player i participates, then i pays p_i to the seller. Third, Nature chooses $\theta \in \Theta$ according to the common prior μ, and messages $(m_1, m_2) \in M_1 \times M_2$ are drawn according to the probability measure $\pi(\theta)$. Fourth, if player i decided to participate, the seller privately sends message m_i to player i; this is independent of whether $-i$ participated or not. Lastly, the players play the simultaneous move game $G(\lambda)$.

The model implicitly assumes that the individual prices are not contingent on the outcomes of the messages. This is without loss of generality, i.e. allowing the prices to be contingent on the messages does not increase the revenue that the seller can get. (See Discussion 5.1.)

3 The Induced Bayesian Game

Taken together, the common prior $\mu \in \Delta(\Theta)$, the information structure $\mathcal{I}$, and the game $G(\lambda)$ induce a Bayesian game. This section analyzes behavior in the Bayesian game.

Fix $\mu \in \Delta(\Theta)$ and $\mathcal{I}$. Write $\Omega := \Theta \times M$ and let $\mathcal{B}$ be its Borel sigma algebra. Let $\phi \in \Delta(\Omega)$ be the unique probability measure such that

$$\phi(D \times E) = \int_{\theta \in D} \pi(\theta)(E)\ d\mu, \tag{1}$$

for each measurable set $D \subseteq \Theta$ and $E \subseteq M$.[2] The measure ϕ is the prior of states and messages that $\mathcal{I}$ induces. Call $P = (\Omega, \mathcal{B}, \phi)$ the **ex-ante probability space induced by** $\mathcal{I}$. Write $\mathbf{M}_i : \Omega \to M_i$ for the projection of Ω onto M_i.

Let $\nu : \Omega \times \mathcal{B} \to [0, 1]$ be a regular conditional probability given the sigma algebra generated by $\mathbf{M}_i$. (Since Ω is Polish, some versions exist; see Durrett (2019).) Construct the belief mapping $\beta_i : M_i \to \Delta(\Theta \times M_{-i})$ so that $\beta_i(m_i) = \text{marg}_{\,\Theta \times M_{-i}} \nu((\theta, m_i, m_{-i}), \cdot)$ for each $(\theta, m_i, m_{-i}) \in \Omega$.[3] Notice that β_i is measurable. (See Lemma A.2.) The belief mapping β_i describes what player i believes about the true state θ and message m_{-i} of the other player, conditional on message m_i. Call $\mathcal{T} = (M_i, \beta_i)_{i \in I}$ the **type structure induced by** $\mathcal{I}$.

The type structure $\mathcal{T}$ and the game $G(\lambda)$ induce a Bayesian game. In the Bayesian game, a **pure strategy** for i is a measurable and bounded function $\sigma_i : M_i \to A_i$. Note that the strict concavity of u_i and convexity of the set A_i imply that each best response is single valued. (See Zimper (2006)). Thus, there is no loss of generality in restricting the analysis to only pure strategy equilibria.

Fix a pure strategy profile $\sigma = (\sigma_1, \sigma_2)$. The player i's expected interim utility of choosing action a_i is given by

$$\overline{u}_i(a_i \mid m_i, \sigma_{-i}) := \int_{\Theta \times M_{-i}} u_i(\theta, a_i, \sigma_{-i}(m_{-i}))\ d\beta(m_i).$$

The interim expected utility for player i, given a message m_i and a strategy profile σ, is given by

$$U_i(m_i|\sigma) := \int_{\Theta \times M_{-i}} u_i(\theta, \sigma_i(m_i), \sigma_{-i}(m_{-i}))\ d\beta(m_i).$$

Definition 3.1. A strategy profile (σ_1^*, σ_2^*) is a **Bayesian Equilibrium** if for each message $m_i \in M_i$, $\sigma^*(m_i) \in \arg\max_{a_i \in A_i} \overline{u}_i(a_i \mid m_i, \sigma_{-i}^*)$.

The solution concept of Bayesian equilibrium requires that each player i chooses an optimal action, given her beliefs β_i, her message m_i and $-i$'s strategy σ_{-i}^*.

[2] Notice that the set of measurable rectangles generate the Borel sigma algebra of the product space. Thus, there is a unique measure $\phi \in \Delta(\Theta \times M)$ that satisfies (1). (See Theorem 1.2.4 in Athreya and Lahiri (2006)).

[3] The mapping β_i may change in terms of the choice of ν. However, these changes would occur only on a set of probability zero.

3.1 Hierarchies of Expectations

The paper characterizes Bayesian equilibrium strategies in terms of a geometric sum of the players' hierarchies of expectations about the state $\theta \in \Theta$. These hierarchies of expectations are defined in the following way. Define the mapping $\overline{\theta}_i^1 : M_i \to \mathbb{R}$ by

$$\overline{\theta}_i^1(m_i) := \int_\Theta \theta \; d\text{marg}_\Theta \; \beta_i(m_i).$$

Note that the the compactness of Θ implies the integral above is well defined and that $\overline{\theta}_i^1(m_i) \in [\min \Theta, \max \Theta]$. Moreover, since $\beta_i : \Omega \to \Delta(\Theta \times M_{-i})$ is measurable, the mapping $\overline{\theta}_i^1$ is measurable. (See Appendix A.4.) Given that $\overline{\theta}_i^k$ is defined for both $i \in I$, inductively define the mapping $\overline{\theta}_i^{k+1} : M_i \to \mathbb{R}$ by

$$\overline{\theta}_i^{k+1}(m_i) := \int_{M_{-i}} \overline{\theta}_{-i}^k(m_{-i}) \; d\text{marg}_{M_{-i}} \; \beta_i(m_i).$$

Note that $\overline{\theta}_i^{k+1}(m_i) \in [\min \Theta, \max \Theta]$ provided that $\overline{\theta}_{-i}^k(m_{-i}) \in [\min \Theta, \max \Theta]$ for each $m_{-i} \in M_{-i}$. Moreover, since $\overline{\theta}_{-i}^k$ and β_i are measurable, then $\overline{\theta}_i^{k+1}$ is measurable. (See Appendix A.4.) Call $\overline{\theta}_i^k(m_i)$ player i's **k-order expectation** of the state conditional on message m_i. The value $\overline{\theta}_i^1(m_i)$ is player i's conditional expectation of the θ given m_i, the value $\overline{\theta}_i^2(m_i)$ is i's conditional expectation of $\overline{\theta}_{-i}^1(m_{-i})$ given m_i, and so on for higher-order expectations.

3.2 Characterization of Bayesian Equilibrium

Theorem 3.1. *Fix an information structure $\mathcal{I}$. The Bayesian game induced by $(G(\lambda), \mu, \mathcal{I})$ has a Bayesian equilibrium (σ_1^*, σ_2^*) where*

(i) $\sigma_i^*(m_i) = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{-\lambda}{2}\right)^{k-1} \overline{\theta}_i^k(m_i)$, *and*

(ii) $U_i(m_i \mid \sigma^*) = \sigma_i^*(m_i)^2$.

Moreover, (σ_1^*, σ_2^*) is unique on a set of probability one.

There are a few remarks worth making about this result. First, Theorem 3.1 states that the Bayesian equilibrium is unique on a set of probability one. Thus, the ex-ante expected payoff that i gets in each information structure does not depend on equilibrium selection. This feature is in contrast to many information design models that involve equilibrium selection. Second, each player's equilibrium strategy depends on her hierarchies of beliefs. However, one feature of this model is that the equilibrium can be written in terms hierarchies of expectations instead of the whole description of the hierarchies of beliefs. Third, the Bayesian equilibrium depends on the hierarchies of expectations in a geometric way. That is, all hierarchies of expectations matter, but the impact decreases the higher the order. Finally, the sign of the effect of each hierarchy depends on whether actions are strategic substitutes or complements. If actions are strategic complements ($\lambda < 0$), the effect of all the hierarchies is positive. This implies that both players increase their action as long as they both "commonly believe" that the state is high. In contrast, if actions are strategic substitutes ($\lambda > 0$), the sign alternates with respect to the parity of the level of the hierarchy. This implies that player i selects a high action if i expects the state is high and players are close to "commonly believing" that they "disagree" about their expectations of the state.

3.3 The Value of Information

This section explores the value each agent assigns to each information structure. It computes the value that i assigns to $\mathcal{I}$ depending on the information that both players receive.

Fix an information structure $\mathcal{I} = (M, \pi)$. Let $(\overline{\theta}_1^k, \overline{\theta}_2^k)_{k\in\mathbb{N}}$ and (σ_1^*, σ_2^*) be the hierarchies of expectations and equilibrium strategies associated to $\mathcal{I}$. Let $P = (\Omega, \mathcal{B}, \phi)$ be the probability space of states and messages that $\mathcal{I}$ induces. Write $\mathbf{\Theta} : \Omega \to \Theta$ and $\mathbf{M}_i : \Omega \to M_i$ for the projection of Ω onto Θ and M_i respectively. Write $\overline{\mathbf{\Theta}}_i^k := \overline{\theta}_i^k \circ \mathbf{M}_i$ and $\boldsymbol{\sigma}_i^* := \sigma_i^* \circ \mathbf{M}_i$ for the random variables that represent the hierarchies of expectations and equilibrium strategies in the space P. Notice that, $\boldsymbol{\sigma}_i^* = \frac{1}{2}\sum_{k=1}^{\infty}(\frac{-\lambda}{2})^{k-1}\ \overline{\mathbf{\Theta}}_i^k$. In addition, $\overline{\mathbf{\Theta}}_i^1 = \mathbb{E}[\mathbf{\Theta} \mid \mathbf{M}_i]$ and $\overline{\mathbf{\Theta}}_i^{k+1} = \mathbb{E}[\overline{\mathbf{\Theta}}_{-i}^k \mid \mathbf{M}_i]$ for each $k \in \mathbb{N}$. (See Lemma A.5.) That is, $\overline{\mathbf{\Theta}}_i^1$ (resp. $\overline{\mathbf{\Theta}}_i^{k+1}$) is a version of the conditional expectation of $\mathbf{\Theta}$ (resp. $\overline{\mathbf{\Theta}}_{-i}^k$) conditional on the sigma algebra generated by $\mathbf{M}_i$. By repeated applications of the law of iterated expectations, $\mathbb{E}[\overline{\mathbf{\Theta}}_i^k] = \mathbb{E}[\mathbf{\Theta}]$, for each $i \in I$ and $k \in \mathbb{N}$.

Call $\mathcal{I} = (M, \pi)$ **silent** for player i, if $|M_i| = 1$, i.e. if agent i always receives the same message. Notice that when agent i decides not to participate in the selling scheme, players face a Bayesian game with an information structure that is silent for i.

Given an information structure $\mathcal{I} = (M, \pi)$, define $\mathcal{I}_{-i} = (M', \pi')$ so that $M_i' = \{*\}$, $M_{-i}' = M_{-i}$ and $\text{marg}_{M_{-i}}\pi'(\theta) = \text{marg}_{M_{-i}}\pi(\theta)$ for each $\theta \in \Theta$. That is, $\mathcal{I}_{-i}$ is the information structure that sends m_{-i} to $-i$, but sends the silent message $*$ to i. Similarly define $\mathcal{I}_{\emptyset}$ as an information structure that is silent to both players. Given a selling scheme $(\mathcal{I}, p_1, p_2)$, if both players participate, players play the Bayesian game induced by $\mathcal{I}$. If $-i$ participates and i does not, they play the Bayesian game induced by $\mathcal{I}_{-i}$, and if no player participates, they play the Bayesian game induced by $\mathcal{I}_{\emptyset}$.

Fix an information structure $\mathcal{I}$, let σ_1^*, σ_2^* be the equilibrium strategies, P the probability space it induces, and $\mathbf{M}_i$ the message of i as a random variable on P. Write $\mathcal{U}_i(\mathcal{I}) := \mathbb{E}[U_i(\mathbf{M}_i \mid \sigma^*)]$ for i's ex-ante expected payoff under equilibrium of the Bayesian game induced by $\mathcal{I}$. Call $V_i(\mathcal{I}) := \mathcal{U}_i(\mathcal{I}) - \mathcal{U}_i(\mathcal{I}_{-i})$ the value that i assigns to $\mathcal{I}$ conditional on $-i$ participating. The value $V_i(\mathcal{I})$ is the maximum payment that player i is willing to pay for participating conditional on $-i$ participating.

Theorem 3.2. *Fix an environment with degree of strategic substitutability λ and an infor-*

mation structure $\mathcal{I}$. Let P be the induced ex-ante probability space and $\boldsymbol{\sigma}_i^*$, $\boldsymbol{\Theta}$ its associated random variables. Then,

(i) $\mathbb{E}[\boldsymbol{\sigma}_i^*] = \frac{1}{2+\lambda}\mathbb{E}[\boldsymbol{\Theta}]$,

(ii) $\mathcal{U}_i(\mathcal{I}) = \mathbb{E}[\boldsymbol{\sigma}_i^{*2}]$,

(iii) If $\mathcal{I}$ is silent for i, then $\mathcal{U}_i(\mathcal{I}) = \left(\frac{1}{2+\lambda}\mathbb{E}[\boldsymbol{\Theta}]\right)^2$, *and*

(iv) $V_i(\mathcal{I}) = Var[\boldsymbol{\sigma}_i^*]$.

Proof. Fix and information structure $\mathcal{I}$. We show each point separately:

(i) By repeatedly using the law of iterated expectations, for each $k \in \mathbb{N}$, $\mathbb{E}[\overline{\boldsymbol{\Theta}}_i^k] = \mathbb{E}[\boldsymbol{\Theta}]$. Therefore,

$$\begin{aligned}\mathbb{E}[\boldsymbol{\sigma}_i^*] &= \sum_{k=1}^{\infty}(\tfrac{-\lambda}{2})^{k-1}\mathbb{E}[\overline{\boldsymbol{\Theta}}_i^k] \\ &= \sum_{k=1}^{\infty}(\tfrac{-\lambda}{2})^{k-1}\mathbb{E}[\boldsymbol{\Theta}] \\ &= \tfrac{1}{2+\lambda}\mathbb{E}[\boldsymbol{\Theta}].\end{aligned}$$

(ii) This statement is a consequence of Theorem 3.1. The result states that the interim utility $U_i(m_i \mid \sigma^*) = \sigma_i^*(m_i)^2$. Thus, $\mathcal{U}_i(\mathcal{I}) = \mathbb{E}[U_i(\mathbf{M}_i \mid \sigma^*)] = \mathbb{E}[\boldsymbol{\sigma}_i^{*2}]$.

(iii) If $\mathcal{I}$ is silent to i, then $|M_i| = 1$, so $\boldsymbol{\sigma}_i^*$ is a constant random variable. Thus, by (i), $\boldsymbol{\sigma}_i^*$ is equal to the constant $\frac{1}{2+\lambda}\mathbb{E}[\boldsymbol{\Theta}]$. Then, by (ii), i's ex-ante expected utility must be $\mathbb{E}[\boldsymbol{\sigma}_i^{*2}] = \left(\frac{1}{2+\lambda}\mathbb{E}[\boldsymbol{\Theta}]\right)^2$.

(iv) Notice that point (i) states that $\mathbb{E}[\boldsymbol{\sigma}_i^*] = \frac{1}{2+\lambda}\mathbb{E}[\boldsymbol{\Theta}]$. Now, since $\mathcal{I}_{-i}$ is silent for i, $\mathcal{U}_i(\mathcal{I}_{-i}) = (\frac{1}{2+\lambda}\mathbb{E}[\boldsymbol{\Theta}])^2$ (see point (iii)). Thus, $\mathcal{U}_i(\mathcal{I}_{-i}) = \mathbb{E}[\boldsymbol{\sigma}_i^*]^2$. In addition point (ii) implies $\mathcal{U}_i(\mathcal{I}) = \mathbb{E}[\boldsymbol{\sigma}_i^{*2}]$. Therefore,

$$\begin{aligned}V_i(\mathcal{I}) &= \mathcal{U}_i(\mathcal{I}) - \mathcal{U}_i(\mathcal{I}_{-i}) \\ &= \mathbb{E}[\boldsymbol{\sigma}_i^{*2}] - \mathbb{E}[\boldsymbol{\sigma}_i^*]^2 \\ &= \mathrm{Var}[\boldsymbol{\sigma}_i^*],\end{aligned}$$

as desired. □

Point (i) of Theorem 3.2 states that i's expected action is independent of the information structure. Thus, from an ex-ante point of view, the players' expected actions are invariant to the information the seller chooses to reveal. This follows by the quadratic nature of the payoffs which imply that the best responses are linear functions. Point (i) has important consequences for optimal policy-making in duopoly games. For instance, suppose a regulator would like to find the information structure that minimizes expected prices or maximizes expected quantity. In this case, Theorem 3.2 states that is not possible to change the player's expected actions by way of selecting information that the players receive.

Point (ii) provides an expression for the expected utility that each player i gets in the Bayesian equilibrium. It states that this value is the expected value of the square of her strategy.

Point (iii) provides the expected value of of any information structure that silent for i. It shows that agent i is completely indifferent among all information structures that are silent to i. In other words, given that i receives no information, i is indifferent about the information that $-i$ gets. This is a consequence of the quadratic nature of the payoffs. In these games, the ex-ante expected action of $-i$ is constant (see point (i)). Thus, given that i receives no information, changes in the information structure for $-i$ only changes the variance of $-i$'s action but not is mean. Consequently, changes in the information structure for $-i$ only change the variance of i's payoff, but not its mean. Thus, i is indifferent among changes in the information $-i$ receives.

Point (iv) states that the value that i assigns to $\mathcal{I}$ is the variance of i's associated equilibrium strategy. This means that i values $\mathcal{I}$ to the extent that it "moves" i's action "in the right way." Moreover, since the variance is non-negative, the value that i assigns to any information structure is non-negative.

Theorem 3.2 characterizes the maximum revenue the seller can get for a fixed information structure $\mathcal{I}$. The price $p_i = \mathrm{Var}[\boldsymbol{\sigma}_i^*]$ is the maximum transfer that player i is willing to pay to participate in the seller's scheme. Thus, for each information structure $\mathcal{I} = (M, \pi)$, the seller's maximum revenue is given by $R(\mathcal{I}) := \mathrm{Var}[\boldsymbol{\sigma}_1^*] + \mathrm{Var}[\boldsymbol{\sigma}_2^*]$. Therefore, the seller's objective is to find the information structure $\mathcal{I}$ that maximizes the sum of the agents' variance of their strategies.

3.4 Benchmark Information Structures

This subsection computes the seller's revenue of information in two benchmark information structures.The first is the information structure that fully reveals the true state θ to both players. I denote this information structure by $\mathcal{I}_{1,2} := (\Theta, \Theta, \pi_{1,2})$ where each message set is Θ, and the message mapping satisfies $\pi_{1,2}(\theta)(\theta_1, \theta_2) = 1$ if and only if $\theta_1 = \theta_2 = \theta$. Notice that for the case $\lambda = 0$, $\mathcal{I}_{1,2}$ maximizes the seller's revenue among all type structures. This follows from the fact that when $\lambda = 0$, each player faces an individual decision problem. Thus, more information is always better and completely revealing the parameter θ maximizes each player's expected payoff. (see Blackwell (1953)).

The second benchmark information structure fully reveals the state to only one player, say player 1, while revealing nothing to player 2. Denote this information structure by $\mathcal{I}_1 := (\Theta, \{*\}, \pi_1)$, where Θ is the message set for player 1, $*$ is a "silent" message for player two and $\pi_1(\theta)(\theta_1, *) = 1$ if and only if $\theta_1 = \theta$. The proposition below provides an expression for the seller's revenue in both benchmark cases.

Proposition 3.1. *Suppose the players face a game $G(\lambda)$. Then,*

(i) $R(\mathcal{I}_1) = \frac{1}{4}\,Var[\mathbf{\Theta}]$, *and*

(ii) $R(\mathcal{I}_{1,2}) = \frac{2}{(2+\lambda)^2}\,Var[\mathbf{\Theta}]$.

Proposition 3.1 provides expressions for the revenue of $\mathcal{I}_1$ and $\mathcal{I}_{1,2}$ as a multiple of $\mathrm{Var}[\mathbf{\Theta}]$. Notice that the variance $\mathrm{Var}[\mathbf{\Theta}]$ is exogenously given by the common prior $\mu \in \Delta(\Theta)$, so it is invariant with respect to the information structure that the seller selects. The more initial uncertainty players have about the state θ, the higher $\mathrm{Var}[\mathbf{\Theta}]$ and the more valuable both information structures become. Figure 3.1 below illustrates the seller's revenue in these two benchmark cases for any degree of strategic substitutability $\lambda \in (-2, 2)$ when $\mathrm{Var}[\mathbf{\Theta}]$ is normalized to 1.

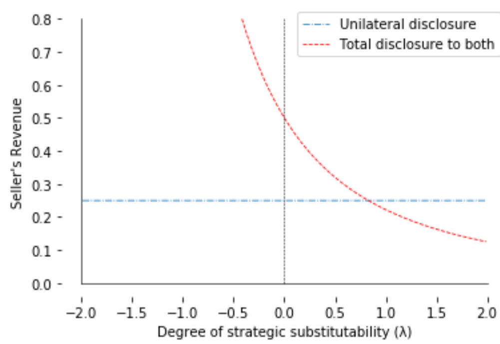

Figure 3.1 Revenue comparison between $\mathcal{I}_1$ and $\mathcal{I}_{1,2}$

There are three important remarks worth making about Figure 3.1. First, the revenue of $\mathcal{I}_1$ is independent of λ. The reason is that, under $\mathcal{I}_1$, player 2 chooses $\boldsymbol{\sigma}_2^* = \left(\frac{1}{2+\lambda}\mathbb{E}[\mathbf{\Theta}]\right)^2$ independently of whether 1 decides to participate or not. Thus, λ, the parameter that governs the relevance of $\boldsymbol{\sigma}_2^*$ to player 1, has no role in how player 1 values $\mathcal{I}_1$.

Second, at $\lambda = 0$, there are no strategic effects. Thus, the revenue of selling the information to two players is twice the revenue of selling it to just one. This means $R(\mathcal{I}_{1,2}) = 2R(\mathcal{I}_1)$ and no informational externalities exist.

Third, at any value $\lambda \neq 0$ the strategic effects leads $R(\mathcal{I}_{1,2}) \neq 2R(\mathcal{I}_1)$. Consider the the case of complements ($\lambda < 0$). Since players actions complement each other, the value of good news (a high realization of θ) increases if the other agent also observes good news. Thus, the value that player 1 assigns to observing θ increases if player 2 also observes θ and a positive informational externality arises. Now consider the case of strategic substitutes ($\lambda > 0$). Since players' actions obstruct each other, the value of good news (a high realization of θ) decreases if the other agent also observes good news. Thus, the value that player 1 assigns to observing θ decreases if player 2 also observes θ and a negative informational externality arises. These effects manifest in the negative slope of $R(\mathcal{I}_{1,2})$ in Figure 2. The higher the degree of sustitutability λ, the higher negative the externality. In fact, when $\lambda > 2(\sqrt{2} - 1) \approx 0.83$, the effect of the externality is so strong that the total value of information is higher when only one agent has access to θ. In other words, for high values of λ the total value of information gets destroyed if information is given to both players instead of to only one. Thus, the seller prefers to offer the information to only player 1 and commits to not sell the information to the other agent. In this way, player 1 accepts to pay a larger amount for $\mathcal{I}_1$ than what both players would pay under $\mathcal{I}_{1,2}$.

4 Characterization of Optimality

This section characterizes the information structures that guarantees the seller's revenue maximization. The key is to use Lemma 4.1 below, which decomposes the value of information of each player in terms of covariances of the players' hierarchies of expectations.

Lemma 4.1. *Let $\mathcal{I}$ be an information structure and $P = (\Omega, \mathcal{B}, \phi)$ be the probability space $\mathcal{I}$ induces. Then,*

(i) $Cov[\overline{\boldsymbol{\Theta}}_i^k, \boldsymbol{\Theta}] = Cov[\overline{\boldsymbol{\Theta}}_i^k, \overline{\boldsymbol{\Theta}}_i^1]$,

(ii) $Cov[\overline{\boldsymbol{\Theta}}_i^k, \overline{\boldsymbol{\Theta}}_{-i}^\ell] = Cov[\overline{\boldsymbol{\Theta}}_i^k, \overline{\boldsymbol{\Theta}}_i^{\ell+1}]$

(iii) $Var[\boldsymbol{\sigma}_i^*] = \frac{1}{2} Cov[\boldsymbol{\sigma}_i^*, \boldsymbol{\Theta}] - \frac{\lambda}{2} Cov[\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_{-i}^*]$, *and*

(iv) $R(\mathcal{I}) = \frac{1}{2} Cov[\boldsymbol{\Theta}, \boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*] - \lambda Cov[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*]$.

The identities from 4.1 are direct consequence of the law of iterated expectations. Figure 4.1 illustrates identities (i) and (ii) by showing a "zig-zag" pattern in the covariance matrix of $\boldsymbol{\Theta}$ and $(\overline{\boldsymbol{\Theta}}_i^k)_{i \in I, k \in \mathbb{N}}$. The identities state that any two entries that are in the same block must be equal. This identities state that the hierarchies of expectations (and thus the equilibrium strategies) cannot be arbitrarily correlated. Identity (iii) uses the first two identities to decompose the value of information for each agent. Identity (iv) uses (iii) to decompose the seller's revenue $\mathcal{I}$ into two components.

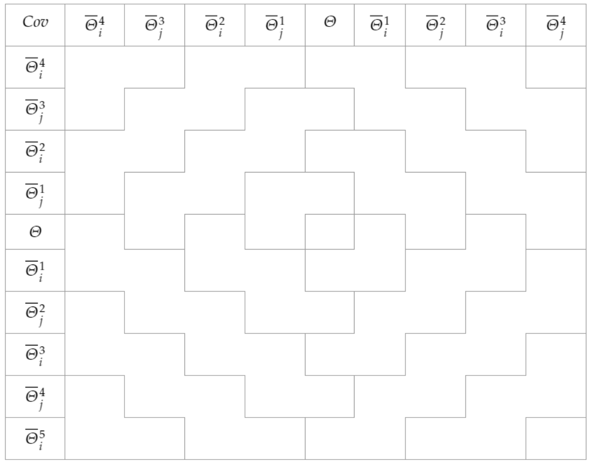

Figure 4.1 Covariance Matrix of $\mathbf{\Theta}$ and its hierarchies of expectations up to the 4-th order.

The first component is given by $\frac{1}{2}\text{Cov}[\mathbf{\Theta}, \boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*]$ and is called direct component of $\mathcal{I}$. This first component is driven by how $\mathcal{I}$ helps players to correlate their actions with the true state θ; the players increase their payoff if they select higher actions when the state is high. The second component is given by $-\lambda\text{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*]$ and is called the strategic component of $\mathcal{I}$. This second component is driven by the correlation that $\mathcal{I}$ induces between players' actions. If actions are strategic complements then the players prefer positive correlation so they can increase the pie. If actions are strategic substitutes, then the players prefer negative correlation to avoid obstructing each other.

Different information structures lead different strategies, and thus, different levels of correlation between the agents' strategies. For instance, under $\mathcal{I}_{1,2}$, both players choose higher strategies for higher values of θ, so their actions are positively correlated. With $\mathcal{I}_1$, player 2's strategy is constant so players' strategies are uncorrelated so the strategic component is zero. The key observation to characterize optimal selling schemes is that the equilibrium conditions imply that if $\text{Cov}[\mathbf{\Theta}, \boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*]$ is high then $\text{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*]$ must be high as well. This implies a trade-off between the first and the second component for the case $\lambda > 0$. On the one hand, the seller wishes to provide more information to the players in order to increase the players' direct component. On the other hand, providing more information leads higher correlation of the players' strategies, decreasing the strategic component. The seller may be willing to sacrifice some of the revenue from the direct component by obfuscating information in a way that decreases the correlation of the agents strategies. Thus, for high values of λ, the seller may need to tune the information structure in a way that finds the optimal balance between providing more information about the state, and inducing opposite actions.

4.1 A Bound on the Seller's Revenue

Proposition 4.1. *Fix an environment with strategic effects given by the parameter* λ*. Then, for each information structure* $\mathcal{I}$*,* $R(\mathcal{I}) \leq \bar{b}(\lambda)\, Var[\mathbf{\Theta}]$*, where*

$$\bar{b}(\lambda) := \begin{cases} \frac{2}{(2+\lambda)^2} & \text{if } \lambda \leq \frac{2}{3} \\ \frac{1}{4\lambda(2-\lambda)} & \text{if } \lambda > \frac{2}{3}. \end{cases}$$

Proposition 4.1 provides an upper bound of the seller's revenue. There are two important remarks about this bound. First, the bound linearly depends on the initial variance of the state. That is, the maximum revenue that the seller can obtain depends on the initial level of uncertainty. The higher uncertainty the more valuable information is for the buyers. Second, the bound has an "U" shape that reaches its minimum at $\lambda = 1$. Figure 4.2 illustrates this bound and compares it with the revenue of $\mathcal{I}_{1,2}$ and $\mathcal{I}_1$.

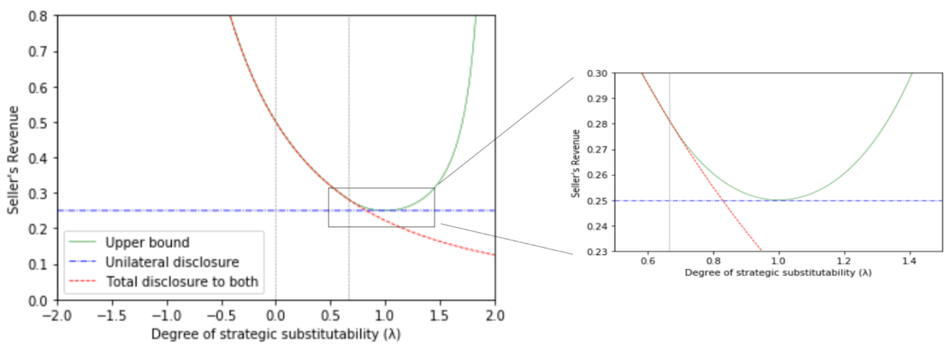

Figure 4.2 Upper-bound of seller's revenue.

Corollary 4.1. *Fix an environment with degree of strategic substitutability* λ*.*

(i) If $\lambda \in (-2, \frac{2}{3})$*, then* $\mathcal{I}_{1,2}$ *maximizes the seller's revenue.*

(ii) If $\lambda = 1$*, then* $\mathcal{I}_1$ *maximizes the seller's revenue.*

Corollary 4.1 is a direct consequence of Proposition 3.1 and Theorem 4.1. With strategic complements ($-2 < \lambda \leq 0$), both components are maximized by $\mathcal{I}_{1,2}$ so the bound is achieved by $\mathcal{I}_{1,2}$. With mild substitutability effects ($0 \leq \lambda \leq \frac{2}{3}$) the direct component dominates the strategic component so the bound is also achieved by $\mathcal{I}_{1,2}$. However, with strong substitutability effects ($\frac{2}{3} < \lambda < 2$) the strategic component is so strong that $\mathcal{I}_{1,2}$ may not maximize revenue. Moreover, if $\lambda = 1$ then $\mathcal{I}_1$ maximizes revenue among all information structures.

4.2 When is the Bound Achievable?

A natural question that arises is whether the upper-bound from Proposition 4.2 is achievable by noisy information structures in the region $\lambda \in (\frac{2}{3}, 1) \bigcup (1, 2)$. The following result provides a characterization of information structures that achieve the upper-bound $\bar{b}$ for high values of λ.

Lemma 4.2. *Fix an environment with degree of strategic substitutability $\lambda \in (\frac{2}{3}, \bar{\lambda})$. Then, maximizing $R(\mathcal{I})$ is equivalent to minimizing $Var[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{1}{2\lambda}\boldsymbol{\Theta}]$. In addition, the following statements are equivalent:*

(i) $R(\mathcal{I}) = \bar{b}(\lambda)\, Var[\boldsymbol{\Theta}]$,

(ii) $Var[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{1}{2\lambda}\boldsymbol{\Theta}] = 0$, *and*

(iii) $Var[\overline{\boldsymbol{\Theta}}_1^1 + \overline{\boldsymbol{\Theta}}_2^1 - \frac{2-\lambda}{\lambda}\boldsymbol{\Theta}] = 0.$

The result provides sufficient conditions for optimality in the region $\lambda \in (\frac{2}{3}, \bar{\lambda})$. Firstly, it founds a dual problem for the revenue maximization problem. The dual is to minimize the variance $\mathrm{Var}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{1}{2\lambda}\boldsymbol{\Theta}]$. Thus, any information structure such that $\mathrm{Var}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{1}{2\lambda}\boldsymbol{\Theta}] = 0$ implies revenue maximization. In addition, the result also provides a simple condition for optimality in terms of only first-order expectations about $\boldsymbol{\Theta}$. This condition states that any information structure such that $\mathrm{Var}[\overline{\boldsymbol{\Theta}}_1^1 + \overline{\boldsymbol{\Theta}}_2^1 - \frac{2-\lambda}{\lambda}\boldsymbol{\Theta}] = 0$ maximizes revenue. It turns out that the existence of such information structure depends on the parameter λ and the shape of the common prior $\mu \in \Delta(\boldsymbol{\Theta})$. On the one hand for each $\lambda \in (\frac{2}{3}, \bar{\lambda})$, there is a prior μ and and information structure $\mathcal{I}$ that achieves the upper-bound. (See Proposition 4.2.) On the other hand, there exist some priors μ such that for almost all values $\lambda \in (\frac{2}{3}, \bar{\lambda})$, there is no information structure $\mathcal{I}$ achieves the upper-bound $\bar{b}$. (See Proposition 4.3.)

Proposition 4.2. *Fix an environment with degree of strategic substitutability $\lambda \in (0, \bar{\lambda})$. Then, there exist a set Θ, a prior $\mu \in \Delta(\Theta)$, and an information structure $\mathcal{I}$ so that $R(\mathcal{I}) = \bar{b}(\lambda)\, Var[\boldsymbol{\Theta}]$.*

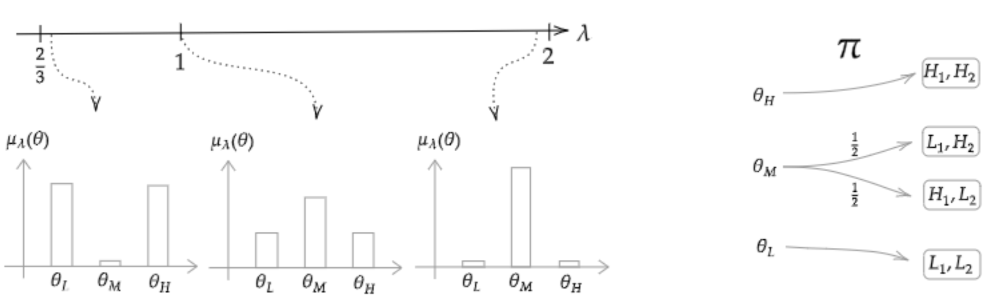

Figure 4.3 Illustration of μ and $\mathcal{I}$ from Proposition 4.2

Proposition 4.2 shows that for each parameter λ, the there exists environments where the bound on revenue can be achieved. The proof constructs an environment with a state space $\Theta = \{\theta_l, \theta_m, \theta_h\}$ where $\theta_l + 1 = \theta_m = \theta_h - 1$ and a prior $\mu \in \Delta(\Theta)$ that depends on λ. In such environment there is a simple information structure $\mathcal{I} = (M, \pi)$ with $M_i = \{L_i, H_i\}$ that achieves the bound. It sends message L_i to each player in case the state is low, sends a high message H_i to each players in case the state is high and send apposite messages when the the state is medium (randomizing with equal probability in both possibilities).

Figure 4.3 illustrates how the prior μ changes with respect to λ. Notice that the higher the parameter λ is, the higher the probability μ assigns to the medium state θ_m and the higher the probability of agents receiving opposite messages. This implies that higher λ leads to 1) lower correlation of the agents' strategies with θ (decreasing the direct component), and 2) lower correlation between the agents' strategies (increasing the strategic component)

Proposition 4.3. *Consider an environment with a parameter $\lambda \in [\frac{2}{3}, \overline{\lambda})$, a state space $\Theta = \{\theta_l, \theta_h\}$ and common prior $\mu \in \Delta(\Theta)$ that assigns uniform probability.*

(i) If $\lambda = \frac{2k}{k+2}$ with $k \in \mathbb{N}$, then exists $\mathcal{I}$ so that $R(\mathcal{I}) = \overline{b}(\lambda)\, Var[\mathbf{\Theta}]$,

(ii) Otherwise, $R(\mathcal{I}) < \overline{b}(\lambda)\, Var[\mathbf{\Theta}]$ for each $\mathcal{I}$.

Proposition 4.3 states that for priors with binary support and uniform probability, is impossible to achieve the upper bound for essentially all parameters $\lambda \geq \frac{2}{3}$. My understanding is that the fact that all the weight of μ is assigned at the extremes creates difficulties on how the seller can use the information to coordinate the agents' strategies.

4.3 When Noise is Optimal?

The results so far show that noisy information structures may dominate the benchmark information structures $\mathcal{I}_1$ and $\mathcal{I}_{1,2}$ when $\lambda \in (\frac{2}{3}, 1) \bigcup (1, \overline{\lambda})$, at least with some priors $\mu \in \Delta(\Theta)$. The question that my current research is trying to answer is whether there is a region of values of λ such that noisy information structures dominate irrespectively of the prior $\mu \in \Delta(\Theta)$. The following conjecture illustrates my guess.

Conjecture 4.1. *There exists a value $\hat{\lambda} > 1$ so that for each $\lambda \in (\hat{\lambda}, \overline{\lambda})$ and each prior $\mu \in \Delta(\Theta)$, there is an information structure $\mathcal{I}$ such that $R(\mathcal{I}) > \max(R(\mathcal{I}_1), R(\mathcal{I}_{1,2}))$.*

4.4 Selling Schemes with Public Communication

In this section I analyse the seller's problem with the assumption that, by exogenous reasons, the seller has the restriction that she can only communicate publicly with the players. Call an information structure $\mathcal{I}$ **public** if $\mathcal{I} = (M, \pi)$ with $M_1 = M_2$ and $\text{Supp}\,(\pi(\theta)) = \{(m_1, m_2) \in M_1 \times M_2 : m_1 = m_2\}$.

Note that given $\mathcal{I}$, the information structure $\mathcal{I}_{-i}$ may not be a public information structure. Thus, in this case the seller cannot provide the information structure $\mathcal{I}_{-i}$ in case i decides not to participate. However, the expression for the value of information from Theorem 3.2 still holds. To see this, assume that the seller provides $\mathcal{I}_\emptyset$ (the information structure that is silent to both players) in case one player or both decide not to participate. Theorem 3.2 implies that $\mathcal{U}_i(\mathcal{I}_\emptyset) = \mathcal{U}_i(\mathcal{I}_{-i})$ since both information structures are silent for i. Thus, $\mathcal{U}_i(\mathcal{I}) - \mathcal{U}_i(\mathcal{I}_{-i}) = \mathcal{U}_i(\mathcal{I}) - \mathcal{U}_i(\mathcal{I}_\emptyset)$, so agent i's value of $\mathcal{I}$ conditional on $-i$ participating is the same as in the previous analysis. Thus, the maximum payment that i is willing to pay for participating is $\text{Var}[\boldsymbol{\sigma}_i^*]$ and previous results hold in the same way as before.

Proposition 4.4. *Fix an environment with degree of strategic substitutability λ, a common prior $\mu \in \Delta(\Theta)$ and suppose the seller is constrained to use public information structures. Then, $\mathcal{I}_{1,2}$ maximizes the seller's revenue.*

Proof. Fix a public information structure $\mathcal{I} = (M, M, \pi)$. Note that $\overline{\theta}_1^k(m) = \overline{\theta}_2^{k'}(m)$ for each message m and each $k, k' \in \mathbb{N}$. That is, all the hierarchies of expectations of both players coincide for each public message m. Then, $\sigma_i(m) = \frac{1}{2+\lambda}\overline{\theta}_i^1(m)$, so $\text{Var}[\boldsymbol{\sigma}_i^*] = \frac{1}{(2+\lambda)^2}\text{Var}[\overline{\boldsymbol{\Theta}}_i]$. In addition, Lemma 4.1 implies $\text{Var}[\overline{\boldsymbol{\Theta}}_i^1] = \text{Cov}[\overline{\boldsymbol{\Theta}}_i^1, \boldsymbol{\Theta}]$ for each player i. Thus,

$$
\begin{aligned}
\text{Var}[\boldsymbol{\Theta}] - \text{Var}[\overline{\boldsymbol{\Theta}}_i^1] &= \text{Var}[\boldsymbol{\Theta}] - 2\text{Cov}[\overline{\boldsymbol{\Theta}}_i^1, \boldsymbol{\Theta}] + \text{Var}[\overline{\boldsymbol{\Theta}}_i^1] \\
&= \text{Var}[\boldsymbol{\Theta} - \overline{\boldsymbol{\Theta}}_i^1] \\
&\geq 0,
\end{aligned}
$$

so $\text{Var}[\overline{\boldsymbol{\Theta}}_i^1] \leq \text{Var}[\boldsymbol{\Theta}]$. Therefore, for each information structure $\mathcal{I}$,

$$
\begin{aligned}
R(\mathcal{I}) &= \text{Var}[\boldsymbol{\sigma}_1^*] + \text{Var}[\boldsymbol{\sigma}_2^*] \\
&= \tfrac{1}{(2+\lambda)^2}\left(\text{Var}[\overline{\boldsymbol{\Theta}}_1^1] + \text{Var}[\overline{\boldsymbol{\Theta}}_2^1]\right) \\
&\leq \tfrac{2}{(2+\lambda)^2}\text{Var}[\boldsymbol{\Theta}] \\
&= R(\mathcal{I}_{1,2}^f).
\end{aligned}
$$

which establishes the result. □

Corollary 4.2. *If $\lambda > 2(\sqrt{2} - 1)$, then public information structures do not maximize revenue. Thus, the seller is better off if she can privately communicate with the buyers.*

The Corollary above follows from the fact that $\mathcal{I}_1$ dominates $\mathcal{I}_{1,2}$ for high levels of λ. Since $\mathcal{I}_{1,2}$ is optimal among the public information structures, it follows that public information structures are not optimal.

5 Discussion

This paper studies markets for information when two uninformed agents play a quadratic game. By the strategic effects of the players' actions, informational externalities arise. In other words, the value of information not only depends on the information itself but also in the information that the other buyer has. This paper shows that the informational externalities have important consequences on how information is spread. The paper shows that in the case actions are strategic complements information flows easily. A third party seller would offer information to both parties. In addition, with the help of a mediation, players themselves would trade information between them. However, in the case that actions are strategic substitutes, frictions to transmit information appear. With strategic substitutes trading information between the players becomes impossible. In addition, if strategic effects are sufficiently strong, a third party seller would opt to obfuscate information by selling information to only one player or by providing noisy signals. The paper closes with a brief discussion of different extensions, comparisons with other papers and questions for future research.

5.1 Selling Schemes with Contingent Prices

The selling schemes discussed in Section 2.2 implicitly assumes that prices cannot be contingent in the signals that the buyers observe. We show that this is without loss of generality. Consider an environment where the price $p_i : M_i \to \mathbb{R}$ is a function contingent in the message. Write $\mathbf{p}_i := p_i \circ \boldsymbol{M}_i$ for the random variable in the space P that represents i's payment. Notice that $\mathbb{E}[\mathbf{p}_i]$ is i's expected payment for participating in the selling scheme. By Theorem 3.3 player i would accept to participate in the selling scheme if and only if $\mathrm{Var}[\boldsymbol{\sigma}_i^*] \geq \mathbb{E}[\mathbf{p}_i]$. That is, player i participates if and only if the value of information exceeds the expected payment. In addition, the seller's expected revenue from a selling scheme is $\mathbb{E}[\mathbf{p}_1] + \mathbb{E}[\mathbf{p}_2]$. Thus, in any selling scheme where agents participate and provide revenue $\mathbb{E}[\mathbf{p}_1] + \mathbb{E}[\mathbf{p}_2]$, there is a selling scheme with fixes prices $p_i = \mathbb{E}[\mathbf{p}_i]$ that achieves the same revenue for the seller.

5.2 Robustness

This paper shows that obfuscating information with noisy signals may be optimal to maximize revenue. However, the optimal noisy information structures are highly sensible to the original prior of the state. Tiny changes in the prior could potentially lead to big losses on the value of information.

However, there are two information structures that are in certain sense "more robust" than any noisy information structure. Notice that $V(\mathcal{I}_1)$ and $V(\mathcal{I}_{1,2})$ depend on the prior

μ only trough the level of uncertainty $\text{Var}(\mathbf{\Theta})$. (See Proposition 3.1.)

My guess is that noisy information structures can potentially do much worse than $\mathcal{I}_{1,2}$ and $\mathcal{I}_1$. That is, for each information structure $\mathcal{I}$ there is a (non degenerated prior) $\mu \in \Delta(\Theta)$ such that

$$R(\mathcal{I}) \leq \max[R(\mathcal{I}_1), R(\mathcal{I}_{1,2})] = \max\{\tfrac{1}{4}, \tfrac{2}{2+\lambda}\}\text{Var}[\mathbf{\Theta}].$$

That is, while a noisy information structure $\mathcal{I}$ may be optimal for a prior μ, $\mathcal{I}$ may be worse than $\mathcal{I}_{1,2}$ or $\mathcal{I}_{1,2}$ for some other prior μ'. Therefore, if the information buyers are "unsure" about the true prior and have min-max preferences they may be willing to pay more for $\mathcal{I}_{1,2}$ or $\mathcal{I}_{1,2}$ than to any other information structure.

5.3 Other Quadratic Forms

The results of optimal selling schemes can be applied to other environments with a more general quadratic form. For instance, suppose the unknown parameter is $\hat{\theta}$ payoff function of agent i is

$$\hat{u}_i(a_i, a_{-i}, \hat{\theta}) = \delta_0 \hat{\theta} a_i - \delta_1 a_i^2 - \delta_2 a_1 a_{-i} - \delta_3 a_i - \delta_4 a_{-i} - f(\hat{\theta}), \tag{2}$$

where $\delta_1 > 0$ and certain function $f : \mathbb{R} \to \mathbb{R}$. This section argues that the selling information model with this payoff function is equivalent to a model with the standard form. This means, is equivalent to analyse the payoff function $\hat{u}_i(a_i, a_{-i}, \theta) = \theta a_i - a_i^2 - \lambda a_i a_{-i}$, by taking the transformation $\theta := \frac{\delta_0 \hat{\theta} - \delta_3}{\delta_1}$ and setting $\lambda = \frac{\delta_2}{\delta_1}$.

To show the equivalence, notice that after normalizing this payoff by δ_1 leads

$$\begin{aligned} u_i(a_i, a_{-i}, \theta) &= \tfrac{\delta_0 \hat{\theta} - \delta_3}{\delta_1} a_i - a_i^2 - \tfrac{\delta_2}{\delta_1} a_i a_{-i} - \tfrac{\delta_4}{\delta_1} a_{-i} - \tfrac{\hat{f}(\theta)}{\delta_1} \\ &= \theta a_i - a_i^2 - \lambda a_i a_{-i} - \tfrac{\delta_4}{\delta_1} a_{-i} - \tfrac{\hat{f}(\theta)}{\delta_1}, \end{aligned}$$

where $\hat{f}(\theta) := f(\delta_1(\frac{\theta}{\delta_0}) + \delta_3)) = f(\hat{\theta})$. Notice that players best response of i is independent of the terms $-\frac{\delta_4}{\delta_1} a_{-i} - \frac{\hat{f}(\theta)}{\delta_1}$. Therefore, for any information structure $\mathcal{I}$, the agents' unique Bayesian equilibrium is still given by (σ_1^*, σ_2^*) from Theorem 3.1. In addition, note that the expected payoff that player i gets under an information structure $\mathcal{I}$ is

$$\begin{aligned} \mathcal{U}_i(\mathcal{I}) &= \mathbb{E}[\mathbf{\Theta} \boldsymbol{\sigma}_i^* - \boldsymbol{\sigma}_i^2 - \lambda \boldsymbol{\sigma}_i^* \boldsymbol{\sigma}_{-i}^* - \tfrac{\delta_4}{\delta_1} \boldsymbol{\sigma}_{-i}^* - \tfrac{1}{\delta_1} g(\mathbf{\Theta})] \\ &= \mathbb{E}[\mathbf{\Theta} \boldsymbol{\sigma}_i^* - \boldsymbol{\sigma}_i^2 - \lambda \boldsymbol{\sigma}_i^* \boldsymbol{\sigma}_{-i}^*] - \tfrac{\delta_4}{\delta_1} \mathbb{E}[\boldsymbol{\sigma}_{-i}^*] - \tfrac{1}{\delta_1} \mathbb{E}[\hat{f}(\mathbf{\Theta})]. \end{aligned}$$

However, notice that $\mathbb{E}[\hat{f}(\mathbf{\Theta})]$ depends only on the prior μ_0 but not in the information structure $\mathcal{I}$. Moreover, the same is true for the term $\mathbb{E}[\boldsymbol{\sigma}_{-i}^*]$. (See Theorem 3.2.) In other words $\mathcal{U}_i(\mathcal{I}) = \mathbb{E}[\mathbf{\Theta} \boldsymbol{\sigma}_i^* - \boldsymbol{\sigma}_i^{*2} - \lambda \boldsymbol{\sigma}_i^* \boldsymbol{\sigma}_{-i}^*] - c$ for certain constant $c \in \mathbb{R}$ and the analysis of the value of information holds in the same way as in Section 3.3.

Example 5.1. This example illustrates a payment function that takes the form (2). In the spirit of the beauty contest, consider the utility function with quadratic form (Morris and

Shin (2002))

$$u_i(a_i, a_{-i}, \theta) = -(1-r)(a_i - \theta)^2 + r(a_i - a_{-i})^2,$$

where $r \in (0,1)$. In this game, agents have incentives to match the action a_i with the state θ and the action of the other player a_{-i}. Notice that expanding this expression leads

$$\begin{aligned} u_i(a_i, a_{-i}, \theta) &= 2(1-r)a_i\theta - a_i^2 + 2ra_ia_{-i} - (1-r)\theta^2 \\ &= \delta_0 a_i \theta - \delta_1 a_i^2 - \delta_2 a_i a_{-i} - f(\theta), \end{aligned}$$

where $\delta_0 = 2(1-r)$, $\delta_1 = -1$, $\delta_2 = -2r$ and $f(\theta) = (1-r)\theta^2$. This, environment is equivalent to an to the former model with a the degree of strategic sustitutability $\lambda = -2r$. Therefore, selling information to both agents maximizes revenue. (See Corollary 4.1.)

5.4 Bayes Correlated Equilibrium

The notion of Bayes correlated equilibrium (BCE) is a solution concept that characterizes all distributions of outcomes that can be generated by a Bayesian equilibrium of any information structure. This solution concept can be used to compute the seller's scheme that maximizes revenue. However, this paper abstracts from it. The reason is that a characterization of BCE outcomes does not reveal what BCE outcomes are attached to a particular information structure. For instance, suppose that a researcher would like to have a prediction of behavior and payoffs for a particular information structure $\mathcal{I}$ she has. The BCE characterization provides a set of outcomes that can be achieved by all information structures but is silent about what outcomes are realized for a particular $\mathcal{I}$. Instead, this paper finds characterizes the unique Bayesian equilibrium for each information structure so a researcher is able to compute the value of information for each agent and each information structure.

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Appendix A Belief Mappings

Fix an information structure $\mathcal{I} = (M, \pi)$ and let $P = (\Omega, \mathcal{B}, \phi)$ the probability space that $\mathcal{I}$ induces. Let $\mathcal{A}_{\mathbf{M}_i}$ be the sigma algebra generated by $\mathbf{M}_i$. Since Ω is Polish, there exist a version of regular conditional probability $\nu : \Omega \times \mathcal{B} \to [0, 1]$ given $\mathcal{A}_{\mathbf{M}_i}$ (see Theorem 5.1.9 Durrett (2019)). Note ν satisfies three properties.

(i) For each $\omega \in \Omega$, $\nu(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{B})$.

(ii) For each $E \in \mathcal{B}$, $\nu(\cdot, E) : \Omega \to [0,1]$ is a $\mathcal{A}_{\mathbf{M}_i}$ measurable.

(iii) For each $E \in \mathcal{B}$, $F \in \mathcal{A}$

$$\int_F \nu(E, \omega)\; d\phi(\omega) = \phi(E \cap F).$$

Define $\hat{\nu} : \Omega \to \Delta(\Omega)$ so that $\hat{\nu}(\omega)(E) = \nu(\omega, E)$ and $\tilde{\nu} : \Omega \to \Delta(\Theta \times M_{-i})$ so that $\hat{\nu}_i(\omega) = \operatorname{marg}_{\Theta \times M_{-i}} \tilde{\nu}(\omega)$. Fix $(\theta^*, m_{-i}^*) \in \Theta \times M_{-i}$ and define $g_i : M_i \to \Omega$ so that $g_i(m_i) = (\theta^*, m_i, m_{-i}^*)$. Define the belief mapping $\beta_i : \Omega \to \Delta(\Theta \times M_{-i})$ by $\beta_i := \tilde{\nu}_i \circ g_i$. Lemma A.2 below shows that β_i is measurable. In addition, Lemma A.3 below shows the choice of (θ^*, m_{-i}^*) does not change the mapping β_i.

Lemma A.1. *The mapping $\hat{\nu} : \Omega \to \Delta(\Omega)$ is measurable.*

Proof. Since Ω is a compact metric space, the Borel sets in the topology of weak convergence is generated by sets of the form $\{\mu \in \Delta(\Omega) : \mu(E) \geq p\}$, for $E \in \mathcal{B}$ and $p \in [0,1]$. (See Gaudard and Hadwin (1989).) Therefore, to show that $\hat{\nu} : \Omega \to \Delta(\Omega)$ is measurable, it suffices to show that $\{\omega \in \Omega : \hat{\nu}(\omega)(E) \geq p\} \in \mathcal{B}$ for each $E \in \mathcal{B}$, $p \in [0,1]$. (See Corollary 4.24 in Aliprantis and Border (2006).) Fix $E \in \mathcal{B}$ and $p \in [0,1]$. Notice that $\nu(\cdot, E) : \Omega \to [0,1]$ is measurable with respect to $\mathcal{A}_{\mathbf{M}_i}$ so is measurable with respect to $\mathcal{B}$ as well. Thus, $\{\omega \in \Omega : \nu(\omega, E) \geq p\} \in \mathcal{B}$ so $\{\omega \in \Omega : \hat{\nu}(\omega)(E) \geq p\} \in \mathcal{B}$ as well. □

Lemma A.2. *The mapping $\beta_i : M_i \to \Delta(\Theta \times M_{-i})$ is measurable.*

Proof. Notice that $\operatorname{proj}_{\Theta \times M_{-i}} : \Omega \to \Theta \times M_{-i}$ is continuous. Then, $\operatorname{marg}_{\Theta \times M_{-i}} : \Delta(\Omega) \to \Delta(\Theta \times M_{-i})$ so that $\operatorname{marg}_{\Theta \times M_{-i}} \mu$ is the image measure of μ under $\operatorname{proj}_{\Theta \times M_{-i}}$ is continuous. (See Theorem 15.14 in Aliprantis and Border (2006).) Since $\hat{\nu}_i$ is measurable (see Lemma A.1), the mapping $\tilde{\nu}_i = \operatorname{marg}_{\Theta \times M_{-i}} \circ \hat{\nu}_i$ is measurable. Since g_i is also measurable, it follows that $\beta_i = \tilde{\nu}_i \circ g_i$ is measurable. □

Lemma A.3. *Fix $\omega, \omega' \in \Omega$. If $\operatorname{proj}_{M_i}(\omega) = \operatorname{proj}_{M_i}(\omega')$, then $\tilde{\nu}(\omega) = \tilde{\nu}(\omega')$*

Proof. Fix $\omega, \omega' \in \Omega$ so that $\operatorname{proj}_{M_i}(\omega) = \operatorname{proj}_{M_i}(\omega')$. Notice that for each $E \in \mathcal{B}$, the mapping $\nu(\cdot, E) : \Omega \to [0,1]$ is measurable with respect to $\mathcal{A}_{\mathbf{M}_i}$. Then, for each $E \in \mathcal{B}$, $\nu(\omega, E) = \nu(\omega', E)$. This implies that $\hat{\nu}(\omega) = \hat{\nu}(\omega')$ and so $\tilde{\nu}(\omega) = \tilde{\nu}(\omega')$. □

A.1 Hierarchies of Expectations

The following lemma implies that all the hierarchies of expectations $(\overline{\theta}_1^k, \overline{\theta}_2^k)_{k \in \mathbb{N}}$ are measurable.

Lemma A.4. *Let $f : \Theta \times M_{-i} \to \mathbb{R}$ be a bounded and measurable mapping. Define $\overline{f} : M_i \to \mathbb{R}$ so that*

$$\overline{f}(m_i) = \int f \; d\beta_i(m_i).$$

Then, $\overline{f}$ is measurable.

Proof. Define $\hat{f} : \Delta(\Theta \times M_{-i}) \to \mathbb{R}$ so that

$$\hat{f}(\mu) = \int f \; d\mu.$$

Since f is bounded and measurable then $\hat{f}$ is well-defined and measurable. (See Theorem 15.14. in Aliprantis and Border (2006)). Therefore, $\overline{f} = \hat{f} \circ \beta_i$ is measurable. (See Lemma A.2.) □

Let $\mathbf{X} : \Omega \to \mathbb{R}$ an integrable random variable on P and ν the fixed regular conditional probability that defines β_i. The mapping $\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i] : \Omega \to \mathbb{R}$ so that

$$\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i](\omega) := \int \mathbf{X}(\omega') \; \nu(\omega, d\omega'),$$

is a version of the conditional expectation of $\mathbf{X}$ given $\mathcal{A}_{\mathbf{M}_i}$ (see Athreya and Lahiri (2006)). That is,

(i) the mapping $\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]$ is $\mathcal{A}_{\mathbf{M}_i}$-measurable, and

(ii) for each $A \in \mathcal{A}_{\mathbf{M}_i}$

$$\int_A \mathbb{E}[\mathbf{X} \mid \mathbf{M}_i] \; d\phi = \int_A \mathbf{X} \; d\phi.$$

Notice that since $\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]$ is $\mathcal{A}_{\mathbf{M}_i}$-measurable, then $\mathbb{E}[\mathbf{\Theta} \mid \mathbf{M}_i](\omega) = \mathbb{E}[\mathbf{\Theta} \mid \mathbf{M}_i](\omega')$ for each $\omega, \omega' \in \Omega$ so that $\mathbf{M}_i(\omega) = \mathbf{M}_i(\omega')$. Denote $\mathbb{E}[\mathbf{\Theta} \mid \mathbf{M}_i = m_i]$ as $\mathbb{E}[\mathbf{\Theta} \mid \mathbf{M}_i](\omega)$ for an element $\omega \in \Omega$ such that $\mathbf{M}_i(\omega) = m_i$.

Lemma A.5. *Write $\overline{\mathbf{\Theta}}_i^k := \overline{\theta}_i^k \circ \mathbf{M}_i$ for each $k \in \mathbb{N}$. Then,*

(i) $\overline{\mathbf{\Theta}}_i^1 = \mathbb{E}[\mathbf{\Theta} \mid \mathbf{M}_i]$, *and*

(ii) $\overline{\mathbf{\Theta}}_i^{k+1} = \mathbb{E}[\overline{\mathbf{\Theta}}_i^k \mid \mathbf{M}_i]$.

Proof. To show (i), fix $\omega \in \Omega$ and write $\mathbf{M}_i(\omega) = m_i$. Notice that

$$\begin{aligned} \overline{\mathbf{\Theta}}_i^1(\omega) &= \overline{\theta}_i^1(m_i) \\ &= \int \theta \; \operatorname{marg}_\Theta \beta_i(m_i) \\ &= \int \mathbf{\Theta}(\omega') \; \nu(\omega, \; d\omega') \\ &= \mathbb{E}[\mathbf{\Theta} \mid \mathbf{M}_i](\omega), \end{aligned}$$

as desired. The proof of (ii) is analogous. □

Appendix B Omitted Proofs

B.1 Proofs from Section 3

Proof of Theorem 3.1

The proof is divided in four steps. The first shows that each type m_i, $\sigma_i^*(m_i) \in A_i$, the second shows that (σ_1^*, σ_2^*) is a Bayesian equilibrium, the third shows uniqueness of Bayesian equilibrium, and the last computes the interim expected utility of each type.

Step 1. Fix $m_i \in M_i$ and write $f_i(m_i) := \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{-\lambda}{2}\right)^{k-1}\overline{\theta}_i^k(m_i)$. We show that for each m_i, $f_i(m_i) \in A_i$.

First consider the first case where $A_i = \mathbb{R}$ and $\lambda \in (-2, 2)$. Notice that $0 \leq \theta_i^k(m_i) \leq \max(\Theta)$ for each $k \in \mathbb{N}$ and $|\frac{\lambda}{2}| < 1$. Thus, $f_i(m_i)$ is a convergent sum so $f_i(m_i) \in A_i$.

Now, consider the second case where $A_i = \mathbb{R}^+$ and $2 < \lambda < \overline{\lambda}$. Note that for each $k \in \mathbb{N}$, $0 \leq \min\Theta \leq \overline{\theta}_i^k(m_i) \leq \max\Theta$. If $\lambda < 0$, then

$$f_i(m_i) = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{-\lambda}{2}\right)^{k-1}\overline{\theta}_i^k(m_i) \geq 0,$$

since each term in the sum is non-negative. If $\lambda \geq 0$ then

$$\begin{aligned} f_i(m_i) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{-\lambda}{2}\right)^{k-1}\overline{\theta}_i^k(m_i) \\ &= \frac{1}{2}\sum_{k\in\mathbb{N}}^{\infty}\left(\frac{-\lambda}{2}\right)^{2k-2}\overline{\theta}_i^{2k-1}(m_i) + \frac{1}{2}\sum_{k\in\mathbb{N}}^{\infty}\left(\frac{-\lambda}{2}\right)^{2k-1}\overline{\theta}_i^{2k}(m_i) \\ &= \frac{1}{2}\sum_{k\in\mathbb{N}}^{\infty}\left(\frac{-\lambda}{2}\right)^{2k-2}\overline{\theta}_i^{2k-1}(m_i) - \frac{\lambda}{4}\sum_{k\in\mathbb{N}}^{\infty}\left(\frac{-\lambda}{2}\right)^{2k}\overline{\theta}_i^{2k}(m_i) \\ &\geq \frac{1}{2}\sum_{k\in\mathbb{N}}^{\infty}\left(\frac{-\lambda}{2}\right)^{2k-2}\ \min(\Theta) - \frac{\lambda}{4}\sum_{k\in\mathbb{N}}^{\infty}\left(\frac{-\lambda}{2}\right)^{2k}\ \max(\Theta) \\ &= \frac{1}{4-\lambda^2}(2\min(\Theta) - \lambda\max(\Theta)). \end{aligned}$$

Then, since $\lambda < \overline{\lambda} = 2\frac{\min\Theta}{\max\Theta}$, it follows that $f_i(m_i) \geq 0$, so $f_i(m_i) \in A_i = \mathbb{R}^+$.

Step 2. Now, I show that $(\sigma_1^*, \sigma_2^*) = (f_1^*, f_2^*)$ is a Bayesian Equilibrium. We shall show that

$$\sigma_i^*(m_i) \in \arg\max_{a_i \in A_i}\left\{\int_{\Theta\times M_{-i}}\left(a_i\ \theta - a_i^2 - \lambda\ a_i\ \sigma_{-i}^*(m_{-i})\right)\ \beta_i(m_i)\right\}.$$

To prove this, it is enough to show that

$$\sigma_i^*(m_i) = \frac{1}{2}\int \theta - \lambda\ \sigma_{-i}^*(m_{-i})\ d\beta_i(m_i). \tag{3}$$

Notice that for each $m_i \in M_i$,

$$\begin{aligned}\sigma_i^*(m_i) &= \tfrac{1}{2}\sum_{k=1}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k-1}\overline{\theta}_i^k(m_i)\\ &= \tfrac{1}{2}\left(\overline{\theta}_i^1(m_i) + \sum_{k=1}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k}\overline{\theta}_i^{k+1}(m_i)\right)\\ &= \tfrac{1}{2}\left(\overline{\theta}_i^1(m_i) - \tfrac{\lambda}{2}\sum_{k=1}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k-1}\overline{\theta}_i^{k+1}(m_i)\right)\\ &= \tfrac{1}{2}\left(\int \theta - \tfrac{\lambda}{2}\sum_{k=1}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k-1}\overline{\theta}_{-i}^k(m_{-i})\ d\beta_i(m_i)\right)\\ &= \tfrac{1}{2}\left(\int \theta - \lambda\ \sigma_{-i}^*(m_{-i})\ d\beta_i(m_i)\right),\end{aligned}$$

where the fourth equality follows from the definition of $\overline{\theta}_i^1(m_i)$ and $\overline{\theta}_i^{k+1}(m_i)$, and the last equality from definition of $\sigma_i^*(m_i)$. Thus, equation (3) is satisfied.

Step 3. Now we compute the interim expected utility of each type m_i under σ^*. Note that

$$\begin{aligned}\overline{u}_i(m_i \mid \sigma^*) &= \int_{\Theta\times M_{-i}} u_i(\theta, \sigma_i^*(m_i), \sigma_{-i}^*(m_{-i})\ d\beta(m_i)\\ &= \int_{\Theta\times M_{-i}} \sigma_i^*(m_i)(\theta - \sigma_i^*(m_i) - \lambda\ \sigma_{-i}^*(m_{-i}))\ d\beta(m_i)\\ &= \sigma_i^*(m_i)\int_{\Theta\times M_{-i}} (\theta - \sigma_i^*(m_i) - \lambda\ \sigma_{-i}^*(m_{-i}))\ d\beta(m_i)\\ &= \sigma_i^*(m_i)\left[\int_{\Theta\times M_{-i}} (\theta - \lambda\ \sigma_{-i}^*(m_{-i}))\ d\beta(m_i) - \sigma_i^*(m_i)\right]\\ &= \sigma_i^*(m_i)^2.\end{aligned}$$

where the fourth equality comes from equation (3).

Step 4. Now, we show that (σ_1^*, σ_2^*) is the unique in a set of probability one. Fix a Bayesian equilibrium $(\hat{\sigma}_1, \hat{\sigma}_2)$. Consider the probability space $P = ((\Theta \times M), \mathcal{B}, \phi)$ induced by $\mathcal{I}$. Write $\hat{\boldsymbol{\sigma}}_i^* := \hat{\sigma}_i^* \circ \mathbf{M}_i$ and recall that $\boldsymbol{\sigma}_i^* := \sigma_i^* \circ \mathbf{M}_i$. To show that the Bayesian equilibrium is unique in a set of probability one, we show that $\mathbb{E}[|\boldsymbol{\sigma}_i^* - \hat{\boldsymbol{\sigma}}_i|] = 0$. Note that (3) implies that $\sigma_i^*(m_i) = \mathbb{E}[\frac{1}{2}\boldsymbol{\Theta} - \frac{\lambda}{2}\boldsymbol{\sigma}_{-i}^* \mid \mathbf{M}_i = m_i]$. Notice,

$$\hat{\sigma}_i(m_i) \in \arg\max_{a_i \in A_i}\left\{\int_{\Theta\times M_{-i}} \left(a_i\ \theta - a_i^2 - \lambda\ a_i\ \hat{\sigma}_{-i}(m_{-i})\right)\ \beta_i(m_i)\right\}.$$

By the first order conditions, $\hat{\sigma}_i(m_i) = \mathbb{E}[\frac{1}{2}\boldsymbol{\Theta} - \frac{\lambda}{2}\hat{\boldsymbol{\sigma}}_{-i} \mid \mathbf{M}_i = m_i]$ if $A_i = \mathbb{R}$. Similarly,

$\hat{\sigma}_i(m_i) = \max[\mathbb{E}[\frac{1}{2}\boldsymbol{\Theta} - \frac{\lambda}{2}\hat{\boldsymbol{\sigma}}_{-i} \mid \mathbf{M}_i = m_i], 0]$ if $A_i = [0, \infty)$. Thus, in either case,

$$\begin{aligned} |\sigma_i^*(m_i) - \hat{\sigma}_i(m_i)| &\leq \left|\mathbb{E}[\tfrac{1}{2}\boldsymbol{\Theta} - \tfrac{\lambda}{2}\boldsymbol{\sigma}_i^* \mid \mathbf{M}_i = m_i] - \mathbb{E}[\tfrac{1}{2}\boldsymbol{\Theta} - \tfrac{\lambda}{2}\hat{\boldsymbol{\sigma}}_i \mid \mathbf{M}_i = m_i]\right| \\ &= \left|\mathbb{E}[\tfrac{\lambda}{2}\boldsymbol{\sigma}_i^* - \tfrac{\lambda}{2}\hat{\boldsymbol{\sigma}}_i \mid \mathbf{M}_i = m_i]\right| \\ &\leq \left|\tfrac{\lambda}{2}\right| \mathbb{E}[|\boldsymbol{\sigma}_i^* - \hat{\boldsymbol{\sigma}}_i| \mid \mathbf{M}_i = m_i]. \end{aligned} \tag{4}$$

Consequently,

$$\begin{aligned} \mathbb{E}\left[|\boldsymbol{\sigma}_i^* - \hat{\boldsymbol{\sigma}}_i|\right] &\leq \left|\tfrac{\lambda}{2}\right| \ \mathbb{E}\left[\mathbb{E}\left[\left|\boldsymbol{\sigma}_{-i}^* - \hat{\boldsymbol{\sigma}}_{-i}\right| \mid \mathbf{M}_i = m_i\right]\right] \\ &= \left|\tfrac{\lambda}{2}\right| \mathbb{E}\left[\left|\boldsymbol{\sigma}_{-i}^* - \hat{\boldsymbol{\sigma}}_{-i}\right|\right]. \end{aligned}$$

So $\mathbb{E}\left[|\boldsymbol{\sigma}_i^* - \hat{\boldsymbol{\sigma}}_i|\right] \leq \left(\frac{\lambda}{2}\right)^2 \ \mathbb{E}\left[|\boldsymbol{\sigma}_i^* - \hat{\boldsymbol{\sigma}}_i|\right]$. Since $|\lambda| < 2$, it follows that $\mathbb{E}\left[|\boldsymbol{\sigma}_i^* - \hat{\boldsymbol{\sigma}}_i|\right] = 0$. Therefore, the Bayesian equilibrium is unique in a set of probability one. □

Proof of Proposition 3.1

First, consider the type structure $\mathcal{I}_1$. Since player 1 observes the true state and player 2 does not observe any signal, then $\overline{\boldsymbol{\Theta}}_1^1 = \boldsymbol{\Theta}$, $\overline{\boldsymbol{\Theta}}_2^1 = \mathbb{E}[\boldsymbol{\Theta}]$. In addition, since both know that player 2 do not observes any signal, $\overline{\boldsymbol{\Theta}}_i^k = \mathbb{E}[\boldsymbol{\Theta}]$ for each $k \in \mathbb{N}$, $k \geq 2$, $i \in I$. Write $c = \frac{1}{2+\lambda}\mathrm{Var}[\boldsymbol{\Theta}]$ and notice that

(i) $\boldsymbol{\sigma}_1^* = \frac{1}{2}\sum_{k\in\mathbb{N}}\left(\frac{-\lambda}{2}\right)^{k-1}\overline{\boldsymbol{\Theta}}_1^k = \frac{1}{2}\boldsymbol{\Theta} - \frac{c}{6}$, and

(ii) $\boldsymbol{\sigma}_2^* = \frac{1}{2}\sum_{k\in\mathbb{N}}\left(\frac{-\lambda}{2}\right)^{k-1}\overline{\boldsymbol{\Theta}}_2^k = c$.

Therefore, $\mathrm{Var}[\boldsymbol{\sigma}_1^*] + \mathrm{Var}[\boldsymbol{\sigma}_2^*] = \mathrm{Var}[\frac{\boldsymbol{\Theta}}{2} - \frac{c}{6}] + \mathrm{Var}[\frac{c}{3}] = \frac{1}{4}\mathrm{Var}[\boldsymbol{\Theta}]$.

Now, consider type structure $\mathcal{I}_{1,2}$. In this case, $\overline{\boldsymbol{\Theta}}_i^k = \boldsymbol{\Theta}$ for each $i \in I$, $k \in \mathbb{N}$. Thus, for each player $i \in I$.

$$\boldsymbol{\sigma}_i^* = \frac{1}{2}\sum_{k\in\mathbb{N}}\left(\frac{-\lambda}{2}\right)^k\boldsymbol{\Theta} = \frac{1}{2+\lambda}\boldsymbol{\Theta}.$$

Therefore, $\mathrm{Var}[\boldsymbol{\sigma}_1^*] + \mathrm{Var}[\boldsymbol{\sigma}_2^*] = 2\mathrm{Var}[\frac{1}{2+\lambda}\boldsymbol{\Theta}] = \frac{2}{(2+\lambda)^2}\mathrm{Var}[\boldsymbol{\Theta}]$. □

B.2 Proofs from Section 4

Lemma B.1. *Fix a probability space P and let $\mathbf{X}$ be a real integrable random variable and $\mathbf{Z}_1$ and $\mathbf{Z}_2$ two other random variables. Let $\overline{\mathbf{X}}_i^1$ be a version of $\mathbb{E}[\mathbf{X} \mid \mathbf{Z}_i]$ and for each $k \in \mathbb{N}$, let $\overline{\mathbf{X}}_i^{k+1}$ be a version of $\mathbb{E}[\overline{\mathbf{X}}_{-i}^k \mid \mathbf{Z}_i]$. Then, for each $k, \ell \in \mathbb{N}$,*

(i) $Cov[\overline{\mathbf{X}}_i^k, \mathbf{X}] = Cov[\overline{\mathbf{X}}_i^k, \overline{\mathbf{X}}_i^1]$, and

(ii) $Cov[\overline{\mathbf{X}}_i^k, \overline{\mathbf{X}}_{-i}^\ell] = Cov[\overline{\mathbf{X}}_i^k, \overline{\mathbf{X}}_i^{\ell+1}]$.

Proof. First we show (i). Note that random variable $\overline{\mathbf{X}}_i^k$ is measurable with respect to the sigma algebra generated by $\mathbf{Z}_i$. Then, $\mathbb{E}[\overline{\mathbf{X}}_i^k \ \mathbf{X} \mid \mathbf{Z}_i] = \overline{\mathbf{X}}_i^k \ \mathbb{E}[\mathbf{X} \mid \mathbf{Z}_i]$ with probability one.

(See Theorem 34.3 in Billingsley (2008).) Thus,

$$\begin{aligned}\mathbb{E}\left[\overline{\mathbf{X}}_i^k\mathbf{X}\right] &= \mathbb{E}\left[\mathbb{E}[\overline{\mathbf{X}}_i^k\mathbf{X}] \mid \mathbf{Z}_i]\right] \\ &= \mathbb{E}\left[\overline{\mathbf{X}}_i^k\mathbb{E}[\mathbf{X} \mid \mathbf{Z}_i]\right] \\ &= \mathbb{E}[\overline{\mathbf{X}}_i^k\overline{\mathbf{X}}_i^1],\end{aligned}$$

where the first equality comes from the law of iterative expectations, and the last equality is given by the definition of $\overline{\mathbf{X}}_i^1$.

Now, by the law of iterated expectations $\mathbb{E}[\overline{\mathbf{X}}_i^1] = \mathbb{E}[\mathbf{X}]$. Thus,

$$\begin{aligned}\text{Cov}[\overline{\mathbf{X}}_i^k, \mathbf{X}] &= \mathbb{E}[\overline{\mathbf{X}}_i^k\mathbf{X}] - \mathbb{E}[\overline{\mathbf{X}}_i^k]\mathbb{E}[\mathbf{X}] \\ &= \mathbb{E}[\overline{\mathbf{X}}_i^k\overline{\mathbf{X}}_i^1] - \mathbb{E}[\overline{\mathbf{X}}_i^k]\mathbb{E}[\overline{\mathbf{X}}_i^1] \\ &= \text{Cov}[\overline{\mathbf{X}}_i^k, \overline{\mathbf{X}}_i^1],\end{aligned}$$

as desired. The proof of (ii) analogous to the proof of (i) by replacing $\mathbf{X}$ with $\overline{\mathbf{X}}_{-i}^{\ell}$. □

Proof of Lemma 4.1

Identities (i) and (ii) follow from Lemma *B*.1.

To show (iii) By using the linearity of the covariance, the first identity implies that

$$\begin{aligned}\text{Cov}[\boldsymbol{\sigma}_i^*, \boldsymbol{\Theta}] &= \tfrac{1}{2}\sum_{k\in\mathbb{N}}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k-1}\ \text{Cov}\left[\overline{\boldsymbol{\Theta}}_i^k, \boldsymbol{\Theta}\right] \\ &= \tfrac{1}{2}\sum_{k\in\mathbb{N}}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k-1}\text{Cov}\left[\ \overline{\boldsymbol{\Theta}}_i^k, \overline{\boldsymbol{\Theta}}_i^1\right] \\ &= \text{Cov}[\boldsymbol{\sigma}_i^*, \overline{\boldsymbol{\Theta}}_i^1].\end{aligned}$$

Analogously, for each $k \in \mathbb{N}$ identity 2 implies $\text{Cov}[\boldsymbol{\sigma}_i^*, \overline{\boldsymbol{\Theta}}_{-i}^k] = \text{Cov}[\boldsymbol{\sigma}_i^*, \overline{\boldsymbol{\Theta}}_i^{k+1}]$. Therefore,

$$\begin{aligned}2\ \text{Var}[\boldsymbol{\sigma}_i^*] &= 2\ \text{Cov}[\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_i^*] \\ &= 2\sum_{k\in\mathbb{N}}\left(\tfrac{-\lambda}{2}\right)^{k-1}\text{Cov}[\boldsymbol{\sigma}_i^*, \overline{\boldsymbol{\Theta}}_i^k] \\ &= \text{Cov}[\boldsymbol{\sigma}_i^*, \overline{\boldsymbol{\Theta}}_i^1] - \tfrac{\lambda}{2}\sum_{k\in\mathbb{N}}\left(\tfrac{-\lambda}{2}\right)^{k-1}\text{Cov}[\boldsymbol{\sigma}_i^*, \overline{\boldsymbol{\Theta}}_i^{k+1}] \\ &= \text{Cov}[\boldsymbol{\sigma}_i^*, \boldsymbol{\Theta}] - \tfrac{\lambda}{2}\sum_{k\in\mathbb{N}}\left(\tfrac{-\lambda}{2}\right)^{k-1}\text{Cov}[\boldsymbol{\sigma}_i^*, \overline{\boldsymbol{\Theta}}_{-i}^k] \\ &= \text{Cov}[\boldsymbol{\sigma}_i^*, \boldsymbol{\Theta}] - \lambda\text{Cov}[\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_{-i}^*],\end{aligned}$$

as desired.

Finally, identity (iv) follows from identity 3 and the definition of $R(\mathcal{I})$. □

Lemma B.2. *Fix an information structure $\mathcal{I}$. Then*

(i) $R(\mathcal{I}) = \frac{1}{4\lambda(2-\lambda)}\,Var[\boldsymbol{\Theta}] - \frac{\lambda}{2-\lambda}\,Var\left[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{1}{2\lambda}\boldsymbol{\Theta}\right]$, *for each* $\lambda \neq 0$, *and*

(ii) $R(\mathcal{I}) = \frac{2}{(2+\lambda)^2}\ \mathit{Var}[\mathbf{\Theta}] - \frac{2-3\lambda}{2+\lambda}\ \mathit{Var}\left[\boldsymbol{\sigma}_1^* - \frac{\mathbf{\Theta}}{2+\lambda}\right] - \frac{2-3\lambda}{2+\lambda}\ \mathit{Var}\left[\boldsymbol{\sigma}_2^* - \frac{\mathbf{\Theta}}{2+\lambda}\right] - \frac{2\ \lambda}{2+\lambda}\,\mathit{Var}\left[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{2\mathbf{\Theta}}{2+\lambda}\right].$

Proof. First we show (i). Fix $\lambda \neq 0$. By Lemma 4.1, $R(\mathcal{I}) = \frac{1}{2}\mathrm{Cov}[\mathbf{\Theta}, \boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*] - \lambda \mathrm{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*]$. This implies

$$2\ \mathrm{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*] = \tfrac{1}{\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_1^*, \mathbf{\Theta}] + \tfrac{1}{\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_2^*, \mathbf{\Theta}] - \tfrac{2}{\lambda}R(\mathcal{I}). \tag{5}$$

Then,

$$\begin{aligned}
\mathrm{Var}\left[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \tfrac{1}{2\lambda}\mathbf{\Theta}\right] &= \tfrac{1}{4\lambda^2}\mathrm{Var}[\mathbf{\Theta}] + \mathrm{Var}[\boldsymbol{\sigma}_1^*] + \mathrm{Var}[\boldsymbol{\sigma}_2^*] + 2\ \mathrm{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*] - \tfrac{1}{\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_1^*, \mathbf{\Theta}] - \tfrac{1}{\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_2^*, \mathbf{\Theta}] \\
&= \tfrac{1}{4\lambda^2}\mathrm{Var}[\mathbf{\Theta}] + R(\mathcal{I}) + 2\ \mathrm{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*] - \tfrac{1}{\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_1^*, \mathbf{\Theta}] - \tfrac{1}{\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_2^*, \mathbf{\Theta}] \\
&= \tfrac{1}{4\lambda^2}\mathrm{Var}[\mathbf{\Theta}] + R(\mathcal{I}) - \tfrac{2}{\lambda}R(\mathcal{I}) \\
&= \tfrac{1}{4\lambda^2}\mathrm{Var}[\mathbf{\Theta}] + \tfrac{\lambda-2}{\lambda}R(\mathcal{I}),
\end{aligned}$$

where the third equality follows from (5). Rearranging this equation shows (i), as desired.

Now we show (ii). Write $C_1 := \frac{2-3\lambda}{2+\lambda}\ \mathrm{Var}\left[\boldsymbol{\sigma}_1^* - \frac{\mathbf{\Theta}}{2+\lambda}\right]$, $C_2 := \frac{2-3\lambda}{2+\lambda}\ \mathrm{Var}\left[\boldsymbol{\sigma}_2^* - \frac{\mathbf{\Theta}}{2+\lambda}\right]$, and $C_3 := \frac{2\ \lambda}{2+\lambda}\mathrm{Var}\left[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{2\mathbf{\Theta}}{2+\lambda}\right]$. First notice that

$$\begin{aligned}
C_1 &= \tfrac{2-3\lambda}{2+\lambda}\ \mathrm{Var}\left[\boldsymbol{\sigma}_1^* - \tfrac{\mathbf{\Theta}}{2+\lambda}\right] \\
&= \tfrac{2-3\lambda}{2+\lambda}\left(\mathrm{Var}[\boldsymbol{\sigma}_1^*] + \tfrac{1}{(2+\lambda)^2}\mathrm{Var}[\mathbf{\Theta}] - \tfrac{2}{(2+\lambda)}\mathrm{Cov}[\boldsymbol{\sigma}_1^*, \mathbf{\Theta}]\right) \\
&= \tfrac{2-3\lambda}{2+\lambda}\ \mathrm{Var}[\boldsymbol{\sigma}_1^*] + \tfrac{2-3\lambda}{(2+\lambda)^3}\mathrm{Var}[\mathbf{\Theta}] - \tfrac{4-6\lambda}{(2+\lambda)^2}\ \mathrm{Cov}[\boldsymbol{\sigma}_1^*, \mathbf{\Theta}].
\end{aligned}$$

Analogously, $C_2 = \frac{2-3\lambda}{2+\lambda}\ \mathrm{Var}[\boldsymbol{\sigma}_2^*] + \frac{2-3\lambda}{(2+\lambda)^3}\mathrm{Var}[\mathbf{\Theta}] - \frac{4-6\lambda}{(2+\lambda)^2}\mathrm{Cov}[\boldsymbol{\sigma}_2^*, \mathbf{\Theta}]$. Therefore,

$$C_1 + C_2 = \tfrac{2-3\lambda}{2+\lambda}R(\mathcal{I}) + \tfrac{4-6\lambda}{(2+\lambda)^3}\mathrm{Var}[\mathbf{\Theta}] - \tfrac{4-6\lambda}{(2+\lambda)^2}\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \mathbf{\Theta}]$$

Now, notice that

$$\begin{aligned}
\mathrm{Var}\left[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \tfrac{2\mathbf{\Theta}}{2+\lambda}\right] &= \mathrm{Var}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*] + \tfrac{4}{(2+\lambda)^2}\mathrm{Var}[\mathbf{\Theta}] - \tfrac{4}{2+\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \mathbf{\Theta}] \\
&= \mathrm{Var}[\boldsymbol{\sigma}_1^*] + \mathrm{Var}[\boldsymbol{\sigma}_2^*] + 2\mathrm{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*] + \tfrac{4}{(2+\lambda)^2}\mathrm{Var}[\mathbf{\Theta}] - \tfrac{4}{2+\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \mathbf{\Theta}] \\
&= R(\mathcal{I}) + 2\mathrm{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*] + \tfrac{4}{(2+\lambda)^2}\mathrm{Var}[\mathbf{\Theta}] - \tfrac{4}{2+\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \mathbf{\Theta}].
\end{aligned}$$

This implies

$$\begin{aligned}
C_3 =& \tfrac{2\lambda}{2+\lambda}\mathrm{Var}\left[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \tfrac{2\boldsymbol{\Theta}}{2+\lambda}\right] \\
=& \tfrac{2\lambda}{2+\lambda}R(\mathcal{I}) + \tfrac{4\lambda}{2+\lambda}\mathrm{Cov}[\boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*] + \tfrac{8\lambda}{(2+\lambda)^3}\mathrm{Var}[\boldsymbol{\Theta}] - \tfrac{8\lambda}{(2+\lambda)^2}\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \boldsymbol{\Theta}] \\
=& \tfrac{2\lambda}{2+\lambda}R(\mathcal{I}) + \tfrac{2}{2+\lambda}\big(\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \boldsymbol{\Theta}] - 2R(\mathcal{I})\big) + \tfrac{8\lambda}{(2+\lambda)^3}\mathrm{Var}[\boldsymbol{\Theta}] - \tfrac{8\lambda}{(2+\lambda)^2}\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \boldsymbol{\Theta}] \\
=& \left(\tfrac{2\lambda}{2+\lambda} - \tfrac{4}{2+\lambda}\right)R(\mathcal{I}) + \tfrac{8\lambda}{(2+\lambda)^3}\mathrm{Var}[\boldsymbol{\Theta}] + \left(\tfrac{2}{2+\lambda} - \tfrac{8\lambda}{(2+\lambda)^2}\right)\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \boldsymbol{\Theta}] \\
=& \tfrac{2\lambda-4}{2+\lambda}R(\mathcal{I}) + \tfrac{8\lambda}{(2+\lambda)^3}\mathrm{Var}[\boldsymbol{\Theta}] + \tfrac{4-6\lambda}{(2+\lambda)^2}\mathrm{Cov}[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*, \boldsymbol{\Theta}],
\end{aligned}$$

where the third equality follows from (5). Therefore,

$$\begin{aligned}
C_1 + C_2 + C_3 &= \left(\tfrac{2-3\lambda}{2+\lambda} + \tfrac{2\lambda-4}{2+\lambda}\right)R(\mathcal{I}) + \left(\tfrac{8\lambda}{(2+\lambda)^3} + \tfrac{4-6\lambda}{(2+\lambda)^3}\right)\mathrm{Var}[\boldsymbol{\Theta}] \\
&= \left(\tfrac{-2-\lambda}{2+\lambda}\right)R(\mathcal{I}) + \tfrac{2\lambda+4}{(2+\lambda)^3}\mathrm{Var}[\boldsymbol{\Theta}] \\
&= -R(\mathcal{I}) + \tfrac{2}{(2+\lambda)^2}\mathrm{Var}[\boldsymbol{\Theta}],
\end{aligned}$$

rearranging this equation shows (*ii*).

□

Proof of Proposition 4.1

Fix an information structure $\mathcal{I}$. First consider the case $\lambda \in (\frac{2}{3}, 2)$. By Lemma B.2,

$$\begin{aligned}
R(\mathcal{I}) &= \tfrac{1}{4\lambda(2-\lambda)}\mathrm{Var}[\boldsymbol{\Theta}] - \tfrac{\lambda}{2-\lambda}\mathrm{Var}\left[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \tfrac{1}{2\lambda}\boldsymbol{\Theta}\right] \\
&\leq \tfrac{1}{4\lambda(2-\lambda)}\mathrm{Var}[\boldsymbol{\Theta}] \\
&= \bar{b}(\lambda)\mathrm{Var}[\boldsymbol{\Theta}].
\end{aligned}$$

Now, consider the case $\lambda \in (-2, \frac{2}{3})$. By Lemma B.2,

$$\begin{aligned}
R(\mathcal{I}) &= \tfrac{2}{(2+\lambda)^2}\ \mathrm{Var}[\boldsymbol{\Theta}] - \tfrac{2-3\lambda}{2+\lambda}\ \mathrm{Var}\left[\boldsymbol{\sigma}_1^* - \tfrac{\boldsymbol{\Theta}}{2+\lambda}\right] - \tfrac{2-3\lambda}{2+\lambda}\ \mathrm{Var}\left[\boldsymbol{\sigma}_2^* - \tfrac{\boldsymbol{\Theta}}{2+\lambda}\right] - \tfrac{2\ \lambda}{2+\lambda}\mathrm{Var}\left[\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \tfrac{2\boldsymbol{\Theta}}{2+\lambda}\right] \\
&\leq \tfrac{2}{(2+\lambda)^2}\ \mathrm{Var}[\boldsymbol{\Theta}] \\
&= \bar{b}(\lambda)\mathrm{Var}[\boldsymbol{\Theta}],
\end{aligned}$$

where the inequality follows from the the fact that $\lambda \leq \frac{2}{3}$ implies $\frac{2-3\lambda}{2+\lambda} \geq 0$. □

Lemma B.3. *Fix a type structure $\mathcal{I}$ and degree of substitutability $\lambda \neq 0$. Then*

$$\tfrac{1}{2}\mathbf{W} = (2-\lambda)\mathbf{U} + \lambda\left(\mathbb{E}[\mathbf{U} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{U} \mid \mathbf{M}_2]\right)$$

where $\mathbf{W} := \overline{\boldsymbol{\Theta}}_1^1 + \overline{\boldsymbol{\Theta}}_2^1 - \frac{2-\lambda}{\lambda}\boldsymbol{\Theta}$, *and* $\mathbf{U} := \boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{1}{2\lambda}\boldsymbol{\Theta}$.

Proof. First Notice that

$$\begin{aligned}
\lambda\ \mathbb{E}[\mathbf{U} \mid \mathbf{M}_i] &= \lambda\ \mathbb{E}[\boldsymbol{\sigma}_i^* + \boldsymbol{\sigma}_{-i}^* - \tfrac{1}{2\lambda}\boldsymbol{\Theta} \mid \mathbf{M}_i] \\
&= \lambda\boldsymbol{\sigma}_i^* +\ \mathbb{E}[\lambda\boldsymbol{\sigma}_{-i}^* - \tfrac{1}{2}\boldsymbol{\Theta} \mid \mathbf{M}_i] \\
&= \lambda\boldsymbol{\sigma}_i^* +\ \mathbb{E}\left[\lambda\sum_{k=1}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k-1}\overline{\boldsymbol{\Theta}}_{-i}^{k} - \tfrac{1}{2}\boldsymbol{\Theta} \mid \mathbf{M}_i\right] \\
&= \lambda\boldsymbol{\sigma}_i^* + \lambda\sum_{k=1}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k-1}\overline{\boldsymbol{\Theta}}_{i}^{k+1} - \tfrac{1}{2}\overline{\boldsymbol{\Theta}}_i^1 \\
&= \lambda\boldsymbol{\sigma}_i^* - 2\sum_{k=1}^{\infty}\left(\tfrac{-\lambda}{2}\right)^{k-1}\overline{\boldsymbol{\Theta}}_{i}^{k} + \tfrac{1}{2}\overline{\boldsymbol{\Theta}}_i^1 \\
&= \lambda\boldsymbol{\sigma}_i^* - 2\boldsymbol{\sigma}_i^* + \tfrac{1}{2}\ \overline{\boldsymbol{\Theta}}_i^1 \\
&= (\lambda-2)\boldsymbol{\sigma}_i^* + \tfrac{1}{2}\ \overline{\boldsymbol{\Theta}}_i^1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(2-\lambda)\mathbf{U} + \lambda\left(\mathbb{E}[\mathbf{U} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{U} \mid \mathbf{M}_2]\right) &= (2-\lambda)(\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \tfrac{1}{2\lambda}\boldsymbol{\Theta}) + (\lambda-2)(\boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^*) + \tfrac{1}{2}(\overline{\boldsymbol{\Theta}}_1^1 + \overline{\boldsymbol{\Theta}}_2^1) \\
&= \tfrac{1}{2}(\overline{\boldsymbol{\Theta}}_1^1 + \overline{\boldsymbol{\Theta}}_2^1) - \tfrac{2-\lambda}{2\lambda}\boldsymbol{\Theta} \\
&= \tfrac{1}{2}\mathbf{W},
\end{aligned}$$

so the desired equality holds. □

Lemma B.4. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random variables on* $(\Omega, \mathcal{B}, \mu)$ *and assume* $\mathbf{X}$ *has finite second moments. Then,* $Cov[\mathbf{X}, \mathbb{E}[\mathbf{X} \mid \mathbf{Y}]] \geq 0$.

Proof. Notice that $\mathrm{Cov}[\mathbf{X}, \mathbb{E}[\mathbf{X} \mid \mathbf{Y}]] = \mathrm{Cov}[\mathbb{E}[\mathbf{X} \mid \mathbf{Y}], \mathbb{E}[\mathbf{X} \mid \mathbf{Y}]]$. (See Lemma B.1.) Thus $\mathrm{Cov}[\mathbf{X}, \mathbb{E}[\mathbf{X} \mid \mathbf{Y}]] = \mathrm{Var}[\mathbb{E}[\mathbf{X} \mid \mathbf{Y}]] \geq 0$. □

Proof of Lemma 4.2

Write $\mathbf{W} := \overline{\boldsymbol{\Theta}}_1^1 + \overline{\boldsymbol{\Theta}}_2^1 - \frac{2-\lambda}{\lambda}\boldsymbol{\Theta}$, and $\mathbf{U} := \boldsymbol{\sigma}_1^* + \boldsymbol{\sigma}_2^* - \frac{1}{2\lambda}\boldsymbol{\Theta}$. Note that Lemma B.2 states

$$R(\mathcal{I}) = \tfrac{1}{4\lambda(2-\lambda)}\mathrm{Var}[\boldsymbol{\Theta}] - \tfrac{\lambda}{2-\lambda}\mathrm{Var}\left[\mathbf{U}\right].$$

Since $\mathrm{Var}[\boldsymbol{\Theta}]$ is exogenous, maximizing $R(\mathcal{I})$ is equivalent to minimizing $\mathrm{Var}[\mathbf{U}]$. In addition, $R(\mathcal{I}) = \overline{b}(\lambda)\mathrm{Var}[\boldsymbol{\Theta}]$ if and only if $\mathrm{Var}[\mathbf{U}] = 0$ so (i) is equivalent to (ii).

We show that (ii) is equivalent to (iii) by showing $\mathrm{Var}[\mathbf{U}] = 0$ if and only if $\mathrm{Var}[\mathbf{W}] = 0$. To show this, notice that Lemma B.3 states

$$\tfrac{1}{2}\mathbf{W} = (2-\lambda)\mathbf{U} + \lambda\left(\mathbb{E}[\mathbf{U} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{U} \mid \mathbf{M}_2]\right) \tag{6}$$

Suppose that $\mathrm{Var}[\mathbf{U}] = 0$. This implies that $\mathbf{U}$ is almost surely constant and so do $\mathbb{E}[\mathbf{U} \mid \mathbf{M}_i]$. Then, $\mathrm{Var}[\mathbf{W}] = 0$ by (6).

Now, suppose $\mathrm{Var}[\mathbf{W}] = 0$. Notice that (6) implies

$$
\begin{aligned}
0 &= \tfrac{1}{4}\mathrm{Var}[\mathbf{W}] \\
&= \mathrm{Var}[(2-\lambda)\mathbf{U} + \lambda\,(\mathbb{E}[\mathbf{U} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{U} \mid \mathbf{M}_2])] \\
&= (2-\lambda)^2\ \mathrm{Var}[\mathbf{U}] + \lambda^2\ \mathrm{Var}[\mathbb{E}[\mathbf{U} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{U} \mid \mathbf{M}_2]] + 2\lambda(2-\lambda)\ \mathrm{Cov}[\mathbf{U}, \mathbb{E}[\mathbf{U} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{U} \mid \mathbf{M}_2]] \\
&\geq (2-\lambda)^2\ \mathrm{Var}[\mathbf{U}],
\end{aligned}
$$

where the last equality follows from $\mathrm{Cov}[\mathbf{U}, \mathbb{E}[\mathbf{U} \mid \mathbf{M}_i] \geq 0$. (See Lemma B.4.) □

Proof of Proposition 4.2

First suppose $\lambda \in (-2, \frac{2}{3})$. If $\lambda \leq \frac{2}{3}$, then $R(\mathcal{I}_{1,2}) = \bar{b}(\lambda)\mathrm{Var}[\mathbf{\Theta}]$ for any prior μ.

Now suppose $\lambda \in (\frac{2}{3}, \bar{\lambda})$. Let $\Theta = \{\theta_L, \theta_M, \theta_H\} \subset \mathbb{R}$ be the set of states so that $\theta_L + 1 = \theta_M = \theta_H - 1$. Define $\mu \in \Delta(\Theta)$ so that $\mu(\theta_M) = \frac{1}{2} - \frac{1-\lambda}{\lambda}$ and $\mu(\theta_L) = \mu(\theta_H) = \frac{1}{4} + \frac{1-\lambda}{2\lambda}$. Write $\mathcal{I} = (M_1, M_2, \pi)$ where $M_i = \{L_i, H_i\}$ and $\pi : \Theta \to \Delta(M_1 \times M_2)$ so that $\pi(\theta_H)(H_1, H_2) = 1$, $\pi(\theta_H)(H_1, L_2) = \frac{1}{2}$, $\pi(\theta_H)(L_1, H_2) = \frac{1}{2}$, and $\pi(\theta_H)(L_1, L_2) = 1$.

Notice that $\beta_i : M_i \to \Delta(\{\theta_L, \theta_m, \theta_h\})$ is given by $\beta_i(L_i) = (\frac{2-\lambda}{2\lambda}, 1 - \frac{2-\lambda}{2\lambda}, 0)$, and $\beta_i(H_i) = (0, 1 - \frac{2-\lambda}{2\lambda}, \frac{2-\lambda}{2\lambda})$. Consequently,

$$
\bar{\theta}_i^1(m_i) = \begin{cases} \theta_m - \frac{2-\lambda}{2\lambda} & \text{if} \quad m_i = L_i \\ \theta_m + \frac{2-\lambda}{2\lambda} & \text{if} \quad m_i = H_i, \end{cases}
$$

This implies that for each $\omega \in \Omega$, $\overline{\mathbf{\Theta}}_1^1(\omega) + \overline{\mathbf{\Theta}}_2^1(\omega) - \frac{2-\lambda}{\lambda}\mathbf{\Theta}(\omega) = \frac{3\lambda-2}{\lambda}\theta_m$. Thus, if follows that $\mathrm{Var}[\overline{\mathbf{\Theta}}_1^1 + \overline{\mathbf{\Theta}}_2^1 - \frac{2-\lambda}{\lambda}\mathbf{\Theta}] = 0$ and $\mathcal{I}$ reaches the upper bound $R(\mathcal{I}) = \bar{b}(\lambda)\mathrm{Var}[\mathbf{\Theta}]$ (see Proposition 4.2). □

Lemma B.5. *Let $\mathbf{X}$ be a Bernoulli random variable in $\{-1, 1\}$ with uniform probability and $\mathbf{M}_1$, $\mathbf{M}_2$ two random variables. Fix $\alpha \in (0, 2)$ so that $\alpha \neq \frac{2}{k}$ for each $k \in \mathbb{N}$. Then,*

$$
\mathit{Var}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{X} \mid \mathbf{M}_2] - \alpha\mathbf{X}] > 0.
$$

Proof. We proceed by contradiction. Suppose $\mathrm{Var}(\mathbb{E}[\mathbf{X} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{X} \mid \mathbf{M}_2] - \alpha\mathbf{X}) > 0$. Then $\mathbb{E}[\mathbf{X} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{X} \mid \mathbf{M}_2] - \alpha\mathbf{X} = c$ for certain $c \in \mathbb{R}$ with probability one. Notice that $\mathbb{E}[\mathbf{X}] = 0$, so by the law of iterated expectations $\mathbb{E}[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = 0$. Thus, $0 = \mathbb{E}[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_1]] + \mathbb{E}[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_2]] - \alpha\mathbb{E}[\mathbf{X}] = c$. Therefore, $c = 0$ and

$$
\mathbb{E}[\mathbf{X} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{X} \mid \mathbf{M}_2] - \alpha\mathbf{X} = 0 \tag{7}
$$

We show that (7) implies that $\mathrm{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = \mathrm{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_{-i}]] = \emptyset$ leading to a contraction.

To show this, first note the following remarks:

(i) $\text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] \subseteq [-1, 1]$.

(ii) If $x \in \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]]$ with $x \neq 1$ then $(x, -\alpha - x) \in \text{Supp}\,[(\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i], \mathbb{E}[\mathbf{X} \mid \mathbf{M}_{-i}])]$.

(iii) If $x \in \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]]$ with $x \neq -1$ then $(x, \alpha - x) \in \text{Supp}\,[(\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i], \mathbb{E}[\mathbf{X} \mid \mathbf{M}_{-i}])]$.

Notice that (i) follows from the fact that $\mathbf{X}$ takes only values in $\{-1, 1\}$. To show (ii), notice that $x \in \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]]$ with $x \neq 1$ implies $(x, -1) \in \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i], \mathbf{X}]$, since $\mathbb{E}[\mathbf{X} \mid \boldsymbol{M}_i] < 1$ only if $\mathbf{X} = -1$ with positive probability. Thus, by (7)

$(x, -\alpha - x, -1) \in \text{Supp}\,[(\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i], \mathbb{E}[\mathbf{X} \mid \mathbf{M}_{-i}], \mathbf{X})]$. Remark (iii) is analogous to (ii). We use $(i), (ii)$, and (iii) to show the following claim.

Claim. For each $k \in \mathbb{N}$ and each i,

(a) $(1 - k\alpha, 1 - (k-1)\alpha) \cap \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = \emptyset$, and

(b) $(-1 + (k-1)\alpha, -1 + k\alpha) \cap \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = \emptyset$.

We show the claim by induction in k. First consider the base case $k = 1$. Suppose that $x \in (1-\alpha, 1) \cap \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]]$. Then, by (ii), $-\alpha - x \in \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_{-i}]]$. However, notice that $-\alpha - x < \alpha - (1-\alpha) = -1$ which contradicts (i). Thus, $(1-\alpha, 1) \cap \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = \emptyset$, so (a) holds. Analogously, (iii) implies $(-1, -1+\alpha) \cap \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = \emptyset$ so (b) holds.

Now, we show the inductive step. Suppose the claim holds for $k \in \mathbb{N}$ and suppose $x \in (1 - (k+1)\alpha, 1 - k\alpha) \cap \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]]$. Then, by (ii), $-\alpha - x \in \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]]$. However, $-\alpha - x \in (-1 + (k-1)\alpha, -1 + k\alpha)$ which contradicts (b). Thus $(1 - (k+1)\alpha, 1 - k\alpha) \cap \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = \emptyset$, so (a) holds for $k+1$. Analogously, (iii) implies $(-1 + k\alpha, -1 + (k+1)\alpha) \cap \text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = \emptyset$, so (b) holds for $k+1$. Therefore, the claim holds as desired.

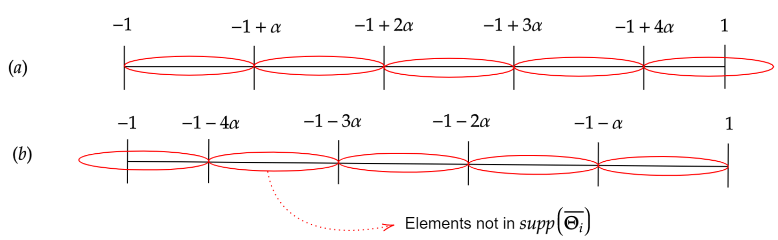

Figure B.1 Illustration of Claim with $\alpha \neq \frac{2}{k}$ for each $k \in \mathbb{N}$.

Finally, notice that if $\alpha = \frac{2}{k}$ for $k \in \mathbb{N}$, then the claim implies that $\text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] \subseteq \{-1, -1+\alpha, ..., 1-\alpha, 1\}$. However, if $\alpha \neq \frac{2}{k}$ for each $k \in \mathbb{N}$ then the claim implies that $\text{Supp}\,[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_i]] = \emptyset$ for each $i \in I$. $\square$

Proof of Proposition 4.3

First we show (i). Suppose for certain $k \in \mathbb{N}$, $\lambda = \frac{2k}{2+k}$. By Theorem 4.2, it is enough to find an information structure such that $\mathrm{Var}[\overline{\mathbf{\Theta}}_1 + \overline{\mathbf{\Theta}}_2 - \frac{2}{k}\mathbf{\Theta}] = 0$. Define $\mathcal{I}_{(k)}$ as follows. Write $M_1 = M_2 = \{0, 1, 2, ..., k\}$ and $\pi : \Theta \to \Delta(M_1 \times M_2)$, so that

$$\pi(\theta_l)(m_1, m_2) = \begin{cases} \frac{1}{2^{k-1}}\binom{k-1}{m_1} & \text{if } m_1 + m_2 = k-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi(\theta_h)(m_1, m_2) = \begin{cases} \frac{1}{2^{k-1}}\binom{k-1}{m_1-1} & \text{if } m_1 + m_2 = k+1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that $\mathbb{P}[\mathbf{\Theta} = \theta_h \mid \mathbf{M}_i = 0] = 0$ since in state θ_h the message 0 is never sent. Similarly, $\mathbb{P}[\mathbf{\Theta} = \theta_h \mid \mathbf{M}_i = k] = 1$ since message k is sent only in state θ_h. In addition, for each $i \in \mathcal{I}$ and each $1 \le m_i \le k-1$,

$$\begin{aligned}
\mathbb{P}[\mathbf{\Theta} = \theta_h \mid \mathbf{M}_i = m_i] &= \frac{\mathbb{P}[\mathbf{\Theta} = \theta_h, \mathbf{M}_i = m_i]}{\mathbb{P}[\mathbf{\Theta} = \theta_l, \mathbf{M}_i = m_i] + \mathbb{P}[\mathbf{\Theta} = \theta_h, \mathbf{M}_i = m_i]} \\
&= \frac{\frac{1}{2^k}\binom{k-1}{m_i-1}}{\frac{1}{2^k}\binom{k-1}{m_i} + \frac{1}{2^k}\binom{k-1}{m_i-1}} \\
&= \frac{\binom{k-1}{m_i-1}}{\binom{k}{m_i}} \\
&= \frac{m_i}{k},
\end{aligned}$$

where the third equality follows from Pascal's triangle identity. This implies that $\overline{\theta}_i^1(m_i) = \left(\frac{k-m_i}{k}\right)\theta_l + \left(\frac{m_i}{k}\right)\theta_h$. Therefore,

$$\overline{\theta}_1(m_1) + \overline{\theta}_2(m_2) = \left(\frac{2k-m_1-m_2}{k}\right)\theta_l + \left(\frac{m_1+m_2}{k}\right)\theta_h.$$

Thus, if $m_1 + m_2 = k-1$, then $\overline{\theta}_1(m_1) + \overline{\theta}_2(m_2) = \frac{k+1}{k}\theta_l + \frac{k-1}{k}\theta_h$. In addition, if $m_1 + m_2 = k+1$, then $\overline{\theta}_1(m_1) + \overline{\theta}_2(m_2) = \frac{k-1}{k}\theta_l + \frac{k+1}{k}\theta_h$. Thus, $\overline{\mathbf{\Theta}}_1 + \overline{\mathbf{\Theta}}_2 = \frac{k-1}{k}(\theta_l + \theta_h) + \frac{2}{k}\mathbf{\Theta}$. Therefore, $\mathrm{Var}[\overline{\mathbf{\Theta}}_1 + \overline{\mathbf{\Theta}}_2 - \frac{2}{k}\mathbf{\Theta}] = 0$.

Now, we show (ii). Fix an information structure $\mathcal{I}$ and write $\mathbf{\Theta} = c_1\mathbf{X} + c_2$ for certain uniform Bernoulli random variable $\mathbf{X}$ that takes values $\{-1, 1\}$. Notice that

$$\mathrm{Var}[\overline{\mathbf{\Theta}}_1 + \overline{\mathbf{\Theta}}_2 - \alpha\mathbf{\Theta}] = c_1^2 \,\mathrm{Var}\left[\mathbb{E}[\mathbf{X} \mid \mathbf{M}_1] + \mathbb{E}[\mathbf{X} \mid \mathbf{M}_2] - \alpha\mathbf{X}\right] > 0$$

where the inequality follows from Lemma B.5. Thus, by Proposition 4.2 it follows that $R(\mathcal{I}) < \overline{b}(\lambda)\mathrm{Var}[\mathbf{\Theta}]$. □

Appendix C Extreme Strategic Effects

This section provides an analysis for extreme values of λ. Here we show that It shows that if $\lambda \le -2$ there may not exist a Bayesian equilibrium. In addition, if $\lambda \ge 2$ then the seller may achieve unlimited revenue by coordinating the players' actions without revealing information about the state θ.

C.1 Extreme Strategic Complementarity

Consider an environment with high strategic complementarity where $\lambda \le 2$. In this case the seller is not able to get revenue by selling information information to the buyers. The condition implies that the slope of the best response in terms of the action of the co-player is higher than one.

Proposition C.1. *Fix an information structure $\mathcal{I}$ and assume $\lambda \le -2$ and $A_i = [0, \infty)$. There exist no Bayesian equilibrium.*

Proof. We proceed by contradiction. Assume (σ_1^*, σ_2^*) is a Bayesian equilibrium. Thus,

$$\hat{\sigma}_i(m_i) \in \arg\max_{a_i \in A_i} \left\{ \int_{\Theta \times M_{-i}} \left(a_i\ \theta - a_i^2 - \lambda\ a_i\ \hat{\sigma}_{-i}(m_{-i})\right)\ \beta_i(m_i) \right\}.$$

Then, by first order conditions, $\sigma_1^*(m_i) \ge \frac{1}{2}\mathbb{E}[\boldsymbol{\Theta} - \lambda \boldsymbol{\sigma}_2^* \mid m_i]$. Since $\frac{-\lambda}{2} \ge 1$, this implies

$$\mathbb{E}[\boldsymbol{\sigma}_i^*] \ge \tfrac{1}{2}\mathbb{E}[\boldsymbol{\Theta}] + \mathbb{E}[\boldsymbol{\sigma}_{-i}^*].$$

Since $\Theta \subset [0, \infty)$ and $\mu \in \Delta(\Theta)$ is assumed to be non degenerated, $\mathbb{E}[\boldsymbol{\Theta}] > 0$. Therefore, $\mathbb{E}[\boldsymbol{\sigma}_i^*] > \mathbb{E}[\boldsymbol{\sigma}_{-i}^*]$ for both $i \in \{1, 2\}$ which leads a contradiction. □

C.2 Extreme Strategic Sustitutability

Consider now the case where $\lambda \ge 2$. In this case the seller can help the agents to achieve high levels of revenue by coordinating their actions instead of given any information about θ. Moreover, if the set of actions is $A_i = \mathbb{R}$, the level of revenue that the seller could achieve is unlimited.

Call an information structure $I = (M, \pi)$ **not informative about the state** if the mapping $\pi : \Theta \to \Delta(M)$ is constant in Θ. An information structure that is not informative about the state may still have value for the agents by coordinating actions between them.

Proposition C.2. *Assume $\lambda \ge 2$ and $A_i = \mathbb{R}$. Then for each $r > 0$ there is a information structure $\mathcal{I}$ that is not informative about the state and a Bayesian equilibrium such that the agents' ex-ante expected utility is at least r.*

Proof. Define $\mathcal{I} = (M, \pi)$ so that $M_i = \{L_i, H_i\}$ and $\pi(\theta)(L_1, H_2) = \frac{2}{2+\lambda}$, $\pi(\theta)(H_1, L_2) = \frac{2}{2+\lambda}$, and $\pi(\theta)(L_1, L_2) = \frac{\lambda-2}{2+\lambda}$ for each $\theta \in \Theta$. The information structure $\mathcal{I}$ induces a belief mapping $\beta_i : M_i \to \Delta(\Theta, M_{-i})$ so that $\text{marg}_{\,\Theta}\beta_i(L_i) = \text{marg}_{\,\Theta}\beta_i(H_i) = \mu$, $\text{marg}_{\,\{L_{-i},H_{-i}\}}\beta_i(H_i) = (1, 0)$, and $\text{marg}_{\,\{L_{-i},H_{-i}\}}\beta_i(L_i) = (\frac{\lambda-2}{\lambda}, \frac{2}{\lambda})$.

Notice that π is constant in θ so it reveals no information about θ. We show that the induced Bayesian game has multiple equilibria parametrized by a constant $c > 0$. Define $\sigma_i^* : M_i \to \mathbb{R}$ so that $\sigma_i^*(H_i) = c$ and $\sigma_i^*(L_i) = \frac{1}{2}\mathbb{E}[\boldsymbol{\Theta}] - \frac{\lambda}{2}c$. We show that σ_i^* constitutes a Bayesian equilibrium. To show this, notice that

$$\sigma_i^*(H_i) = \tfrac{1}{2}\mathbb{E}[\boldsymbol{\Theta}] - \tfrac{\lambda}{2}\left[\tfrac{\lambda-2}{\lambda}\sigma_{-i}^*(H_{-i}) + \tfrac{2}{\lambda}\sigma_{-i}^*(L_{-i})\right], \text{ and}$$
$$\sigma_i^*(L_i) = \tfrac{1}{2}\mathbb{E}[\boldsymbol{\Theta}] - \tfrac{\lambda}{2}\left[\sigma_{-i}^*(H_{-i})\right],$$

which implies that for each $m_i \in M_i$,

$$\sigma_i^*(m_i) = \int_{\Theta \times M_i} \tfrac{1}{2}\theta - \tfrac{\lambda}{2}\hat{\sigma}_{-i}(m_{-i})\ d\beta_i(m_i).$$

Consequently,

$$\sigma_i^*(m_i) \in \arg\max_{a_i \in A_i} \left\{ \int\limits_{\Theta \times M_{-i}} \left(a_i\ \theta - a_i^2 - \lambda\ a_i\ \hat{\sigma}_{-i}(m_{-i})\right)\ d\beta_i(m_i) \right\},$$

so (σ_1^*, σ_2^*) is a Bayesian equilibrium. In addition, $U_i(m_i \mid \sigma^*) = \sigma_i^*(m_i)^2$. Thus, for each $r > 0$ there is $c > 0$ so that $\sigma_i^*(m_i)^2 > r$ for each message m_i. Thus, there exists a value $c > 0$ so that i's ex-ante utility is higher than $r > 0$. □